Degrees of Freedom Optimal Transmission for the Two-Cluster MIMO Multi-way Relay Channel

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***Abstract*—This paper investigates the degrees of freedom (DoF) of the two-cluster multi-way relay channel. Two cases are investigated: the case when there are 2 users in each cluster with arbitrary number of antennas and the case when there are 3 users in each cluster in the symmetric setting, i.e., all the users have the same number of antennas. For the 2-user case, a DoF upper bound is derived based on cut set bound by allowing user cooperation between clusters. Conditions when the DoF upper bound can be achieved using signal space alignment are established based on the relative number of antennas between the users and the relay. For the 3-user case, a new DoF upper bound is derived using genie-aided approach and channel enhancement. The DoF upper bound can be achieved using signal space alignment for several scenarios of interests. The results point out the insight that increasing the number of users in each cluster cannot provide a further DoF gain compared to the 2-user case, when the relay has limited number of antennas.**

I. Introduction

Relaying is a fundamental operation in wireless communications, which can provide capacity gain and robustness against channel uncertainty. The multi-way relay channel [1] characterizes another important role of the relay, i.e., its role in facilitating the exchange of information between the users, and is an important building block for wireless networks. This channel provides a general model that captures relay-aided multicast communication [1]. While the model is quite general, the complex signal interactions render the capacity region of the multi-way relay channel a challenging open problem. Capacity bounds of the multi-way relay channel and special cases have been studied in [1]–[5] and the references therein.

As is done often for multiuser information theoretic models, one can hope to gain insights into optimal transmission strategies for the Gaussian multi-way relay channel by focusing on the capacity characterization in high signal-to-noise ratio (SNR) regime, i.e., the degrees of freedom (DoF). For example, interference alignment has been shown to be the key technique to achieve the optimal DoF for several class of wireless networks [6], [7]. In turn, this concept has proved useful for finite SNR scenarios as well and is extensively utilized to design transmission strategies for the interference networks [8]. The DoF for a multi-way type relay channel is first studied in [9], where the DoF of the Y channel, which is a special case of the multi-way relay channel that consists of 3 users in one cluster and a relay node, is established using signal space alignment. The signal space alignment is further utilized to design transmission strategies for the K-user Y channel in reference [10].

In this paper, we consider the Gaussian multi-antenna multi-way relay channel with two clusters. In previous work [11], we have reported the DoF of the special case when each cluster has 2 users and all the users have identical number of antennas. In this work, we consider two general cases: the case with 2 users in each cluster with arbitrary number of antennas and the case with 3 users in each cluster with the same number of antennas. For the 2-user case with arbitrary number of antennas, we derive DoF upper bounds using cut set bound and identify several cases when the optimal DoF can be established using signal space alignment. For the two-cluster multi-way relay channel with 3 users in each cluster, cut set bounds can not provide a tight upper bound. To address this issue, we utilize a novel genie-aided approach with enhancement of the received signals at the users to derive a new DoF upper bound. We also identify several cases when the newly derived DoF upper bound can be achieved using signal space alignment.

By way of this, we provide the first DoF result for the multi-way relay channel with more than one cluster and more than 2 users in each cluster. While the DoF of the MIMO multi-way relay channel with arbitrary number of users and clusters is yet open, the present result provides us with the insight that increasing the number of users at each cluster cannot provide a DoF gain when the number of antennas at the relay is limited.

II. System Model

The Gaussian MIMO multi-way relay channel with two clusters is shown in Fig. 1. We label the users in cluster 1 as user $i = 1, 2, \cdots, L$, and the users in cluster 2 as user $j = L+1, L+2, \cdots, 2L$. User k, $k = 1, \cdots, 2L$ is assumed to have M_k antennas. The relay is assumed to have N antennas.

In cluster 1, user i has message $W_{pi}, p = 1, 2, \cdots, L, p \neq i$, for each user in cluster 1. Similarly, in cluster 2, user j has a message $W_{qj}, q = L+1, L+2, \cdots, 2L, q \neq j$, for each user in cluster 2. We denote $\overline{W}_k$ as the message set of user k. It is assumed that the users can only communicate through the relay and no direct links exist between any pairs of users. All the nodes in the network are assumed to be full duplex. The transmitted signal from user k in slot t is denoted as $\mathbf{X}_k(t) \in$

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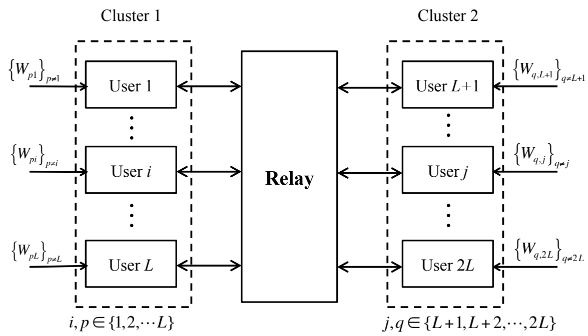

Fig. 1. The multi-way relay channel with two clusters.

$\mathbb{C}^{M_k}$, where

$$\mathbf{X}_k(t) = f_k(\overline{W}_k, \mathbf{Y}_k^{t-1}), \tag{1}$$

where $\mathbf{Y}_k^{t-1} = [\mathbf{Y}_k(1), \cdots, \mathbf{Y}_k(t-1)]$.

The received signal at the relay in slot t is denoted as $\mathbf{Y}_R(t) \in \mathbb{C}^N$, where

$$\mathbf{Y}_R(t) = \sum_{k=1}^{2L} \mathbf{H}_{Rk}\mathbf{X}_k(t) + \mathbf{Z}_R(t). \tag{2}$$

At slot t, the transmitted signal $\mathbf{X}_R(t) \in \mathbb{C}^N$ from the relay is a function of its received signals from slot 1 to slot $t-1$, i.e.,

$$\mathbf{X}_R(t) = f(\mathbf{Y}_R^{t-1}). \tag{3}$$

The received signal at user k in slot t is denoted as $\mathbf{Y}_k(t) \in \mathbb{C}^{M_k}$, where

$$\mathbf{Y}_k(t) = \mathbf{H}_{kR}\mathbf{X}_R(t) + \mathbf{Z}_k(t). \tag{4}$$

In the above expressions, $\mathbf{H}_{kR} \in \mathbb{C}^{M_k \times N}$ and $\mathbf{H}_{Rk} \in \mathbb{C}^{N \times M_k}$ are the channel matrices from the relay to user k and the channel matrices from user k to the relay, respectively. It is also assumed that each entry of the channel matrices is drawn independently from a continuous distribution, which guarantees that the channel matrices are full rank almost surely. $\mathbf{Z}_k(t) \in \mathbb{C}^{M_k}, \mathbf{Z}_R(t) \in \mathbb{C}^N$ are additive white Gaussian noise vectors with zero mean and identity covariance matrices. The transmitted signals from the users and the relay satisfy the following power constraints:

$$E\left[\mathrm{tr}\left(\mathbf{X}_k(t)\mathbf{X}_k(t)^\dagger\right)\right] \le P, \tag{5}$$

$$E\left[\mathrm{tr}\left(\mathbf{X}_R(t)\mathbf{X}_R(t)^\dagger\right)\right] \le P. \tag{6}$$

We assume the rate of message W_{pi} and W_{qj} is $R_{pi}(P)$ and $R_{qj}(P)$ under power constraint P. We define $\mathcal{C}(P)$ as the set of all achievable rate tuples $\{R_{pi}(P), R_{qj}(P)\}$ for power constraint P. The degrees of freedom is defined as

$$DoF = \lim_{P\to\infty} \frac{R_{\sum}(P)}{\log(P)}, \tag{7}$$

where

$$R_{\sum}(P) = \sup_{\{R_{pi}(P), R_{qj}(P)\} \in \mathcal{C}(P)} \sum_{i=1}^{L} \sum_{\substack{p=1 \\ p\neq i}}^{L} R_{pi}(P) + \sum_{j=L+1}^{2L} \sum_{\substack{q=L+1 \\ p\neq j}}^{2L} R_{qj}(P) \tag{8}$$

is the sum capacity under power constraint P.

In this paper, we focus on two cases of the two-cluster multi-way relay channel, as described in the following sections.

III. Two-User Case with Arbitrary Number of Antennas

In this section, we focus on the two-cluster multi-way relay channel with 2 users in each cluster, and the users have arbitrary number of antennas. From the channel model described in the previous section, there are four users in the channel, where cluster 1 contains user 1 and user 2, and cluster 2 contains user 3 and user 4. Without loss of generality we assume that $M_1 \ge M_2$ and $M_3 \ge M_4$.

A. Upper Bound for DoF

Proposition 1: For the two-cluster multi-way relay channel with 2 users in each cluster shown in Fig. 1, the DoF is upper bounded by

$$DoF \le 2\min\{M_2 + M_4, N\}. \tag{9}$$

Proof: The DoF upper bound can be obtained by allowing users to cooperate across clusters and applying the cut set bound. The details are omitted due to space constraints. ■

B. Achieving the DoF Upper Bound

In this section, we design transmission strategies and calculate the achievable DoF. We further identify several cases when the DoF upper bound in equation (9) is achievable.

Theorem 1: For the two-cluster multi-way relay channel with 2 users in each cluster, the optimal DoF can be characterized as follows:

1) $N \le M_2 + M_4$: $DoF = 2N$ if one of the following conditions is satisfied

- *Case 1*: $N \le \max\{M_2, M_4\}$;
- *Case 2*: $N > \max\{M_2, M_4\}$
  - $N \le M_1$ and $N \le M_3$;
  - $M_1 \le N \le M_3$ and $M_1 + M_2 + M_4 \ge 2N$;
  - $M_3 \le N \le M_1$ and $M_3 + M_4 + M_1 \ge 2N$;
  - $N \ge M_1$, $N \ge M_3$ and $M_1 + M_2 + M_3 + M_4 \ge 3N$;

2) $N > M_2 + M_4$: $DoF = 2(M_2 + M_4)$ if one of the following conditions is satisfied

- *Case 1*: $N \ge 2(M_2 + M_4)$;
- *Case 2*: $N < 2(M_2 + M_4)$, $N \le M_1$ and $N \le M_3$.

To prove this theorem, we briefly describe the transmission strategies that can achieve the DoF upper bounds corresponding to the above scenarios.

1) $N \leq M_2 + M_4$: Under this condition, the DoF upper bound in equation (9) reduces to

$$DoF \leq 2N. \tag{10}$$

We further consider the following cases:

Case 1: $N \leq \max\{M_2, M_4\}$.

For this case, the relay always has less antennas than both users in at least one of the two clusters. The DoF $2N$ can be achieved by only allowing the users in the cluster with more antennas than the relay to transmit, which gives us a two-way relay channel. The functional-decode-and-forward (FDF) strategy can thus achieve the DoF $2N$.

Case 2: $N > \max\{M_2, M_4\}$. To establish DoF results for this case, let us first consider the following Lemma:

Lemma 1: For matrices $\mathbf{H}_1 \in \mathbb{C}^{p\times q_1}$ and $\mathbf{H}_2 \in \mathbb{C}^{p\times q_2}$, which have full rank almost surely, the following conditions can be established. Without loss of generality we assume $q_1 \geq q_2$.

Condition 1: If $p \geq q_1 \geq q_2$ and $q_1 + q_2 > p$, then there exist $q_1 + q_2 - p$ non-zero linearly independent vectors $\mathbf{v}_i$ almost surely such that we can find another two sets of linearly independent vectors $\mathbf{u}_i$ and $\mathbf{w}_i$, $i = 1, \cdots, q_1 + q_2 - p$ such that

$$\mathbf{v}_i = \mathbf{H}_1\mathbf{u}_i = \mathbf{H}_2\mathbf{w}_i. \tag{11}$$

Condition 2: If $q_1 \geq p \geq q_2$, then there exist q_2 non-zero linearly independent vectors $\mathbf{v}_i$ almost surely such that we can find another two sets of linearly independent vectors $\mathbf{u}_i$ and $\mathbf{w}_i$, $i = 1, \cdots, q_2$ such that

$$\mathbf{v}_i = \mathbf{H}_1\mathbf{u}_i = \mathbf{H}_2\mathbf{w}_i \tag{12}$$

Proof: The proof of this lemma is given in Appendix A. ■

With this lemma at hand, we consider the following cases:

$N \leq M_1$ and $N \leq M_3$: From Condition 2 in *Lemma 1*, if we set $\mathbf{H}_1 = \mathbf{H}_{R1}$ and $\mathbf{H}_2 = \mathbf{H}_{R2}$, we can see that for user 1 and user 2 in cluster 1, they can find M_2 non-zero linearly independent vectors $\mathbf{v}_{1i}, \mathbf{u}_{1i}$ and $\mathbf{u}_{2i}$ such that

$$\mathbf{H}_{R1}\mathbf{u}_{1i} = \mathbf{H}_{R2}\mathbf{u}_{2i} = \mathbf{v}_{1i}. \tag{13}$$

This means that user 1 and user 2 can share M_2-dimensional space at the relay. Following the same argument, we can see that user 3 and user 4 in cluster 2 also share M_4-dimensional space at the relay, i.e., they can find M_4 non-zero linearly independent vectors $\mathbf{v}_{2i}, \mathbf{u}_{3i}$ and $\mathbf{u}_{4i}$ such that

$$\mathbf{H}_{R3}\mathbf{u}_{3i} = \mathbf{H}_{R4}\mathbf{u}_{4i} = \mathbf{v}_{2i}. \tag{14}$$

Since we have $M_2 + M_4 \geq N$, the users in cluster 1 can choose M_2' vectors out of the vectors $\mathbf{v}_{1i}$, and the users in cluster 2 can choose M_4' vectors out of the vectors $\mathbf{v}_{2i}$, such that $M_2' + M_4' = N$, as their target signal directions at the relay.

Based on the above analysis, we can construct the transmission scheme as follows: User 1 and user 2 send M_2' independent data streams d_{1i} and d_{2i} along the directions $\mathbf{u}_{1i}$ and $\mathbf{u}_{2i}$, respectively. User 3 and user 4 send M_4' independent data streams d_{3i} and d_{4i} along the directions $\mathbf{u}_{3i}$ and $\mathbf{u}_{4i}$, respectively. We have

$$\mathbf{X}_k = \sum_{i=1}^{M_2'} \mathbf{u}_{ki} d_{ki}, k = 1, 2, \tag{15}$$

$$\mathbf{X}_k = \sum_{i=1}^{M_4'} \mathbf{u}_{ki} d_{ki}, k = 3, 4. \tag{16}$$

The received signal at the relay is

$$\mathbf{Y}_R = \sum_{k=1}^{4} \mathbf{H}_{Rk}\mathbf{X}_k \tag{17}$$

$$= \sum_{i=1}^{M_2'} \mathbf{v}_{1i}(d_{1i} + d_{2i}) + \sum_{i=1}^{M_4'} \mathbf{v}_{2i}(d_{3i} + d_{4i}) \tag{18}$$

The relay can then decode $d_{1i} + d_{2i}$ and $d_{3i} + d_{4i}$.

Next the relay needs to transmit $d_{1i} + d_{2i}$ to user 1 and user 2 and in the meantime transmit $d_{3i} + d_{4i}$ to user 3 and user 4. We let the users apply a receiver-side filter $\mathbf{w}_{ki},\ k = 1, 2, 3, 4$ such that

$$(\mathbf{w}_{1i})^T\mathbf{H}_{1R} = (\mathbf{w}_{2i})^T\mathbf{H}_{2R} = \mathbf{g}_{1i}^T,$$
$$(\mathbf{w}_{3i})^T\mathbf{H}_{3R} = (\mathbf{w}_{4i})^T\mathbf{H}_{4R} = \mathbf{g}_{2i}^T,$$

which makes the users in one cluster appear to be the same user to the relay.

We can see that finding the vectors $\mathbf{w}_{ki}$ for the receiver-side filter is the dual problem of finding the vectors $\mathbf{u}_{ki}$ for the transmitter-side beamforming. Following the same argument, the users in cluster 1 (2) can find M_2' (M_4') such vectors. The relay can then use zero-forcing to broadcast $d_{1i} + d_{2i}$ to user 1 and user 2 along target directions $\mathbf{g}_{1i}$ and in the meantime broadcast $d_{3i} + d_{4i}$ to user 3 and user 4 along target directions $\mathbf{g}_{2i}$.

The users can now subtract their own contribution to decode the intended data streams. Therefore we can achieve the DoF $2N$.

For the cases $M_1 \leq N \leq M_3$, $M_3 \leq N \leq M_1$ and $N \geq M_1, N \geq M_3$, the DoF $2N$ can be achieved in a similar fashion and the details are omitted.

2) $N > M_2 + M_4$: Under this setting, the DoF upper bound in equation (9) reduces to

$$DoF \leq 2(M_2 + M_4). \tag{19}$$

Case 1: $N \geq 2(M_2 + M_4)$.

The DoF upper bound can be easily achieved for this case. Since we have $M_1 \geq M_2$, $M_3 \geq M_4$, we can let user 1 use only M_2 of its antennas and let user 3 use only M_4 of its antennas to transmit. The relay can decode all the messages and broadcast the messages to the intended users since it has sufficient spatial dimensions.

Case 2: $N \leq M_1, N \leq M_3$.

From *Lemma 1*, this condition implies that the users in cluster 1 share M_2-dimensional signal space at the relay, and the users in cluster 2 share M_4-dimensional signal space at the relay. It can be verified that the M_2 dimensional signal space shared by the users in cluster 1 has no intersection with the M_4 dimensional signal space shared by the users in cluster 2. Following the transmission scheme we used for previous cases, we let the users in cluster 1 to transmit M_2 independent data streams, and let the users in cluster 2 to transmit M_4 independent data streams, both along their shared dimensions of the signal space at the relay. The relay can decode the sum of the messages and forward to the users using zero-forcing with proper receiver-side filtering at the users. The users can then decode the intended data streams by subtracting their own contribution from the received signals to achieve the DoF $2(M_2 + M_4)$.

Remark 1: For the other cases, the DoF upper bound cannot be achieved using the signal space alignment schemes. The optimal DoF remains unknown.

IV. THREE-USER SYMMETRIC CASE

In this section, we consider the two-cluster multi-way relay channel with 3 users in each cluster in the symmetric setting, i.e., the users are all assumed to have M antennas. From the definition in Section II, we have $L = 3$ for this case.

A. Outerbound on DoF

The cut set bound we used to derive the DoF upper bound for the 2-user case yields a DoF upper bound $3\min\{2M, N\}$ for the 3-user case, which is not tight. To address this issue, we utilize the technique in [12] to enhance the received signal of one of the users, such that it can decode the messages intended for the other users when provided with proper genie information, to improve the upper bound.

Proposition 2: For the two-cluster multi-way relay channel with 3 users in each cluster, the DoF upper bound for the symmetric setting when all users have M antennas and the relay has N antennas is

$$DoF \leq 2\min\{3M, N\}. \tag{20}$$

Proof: We first consider the case when $M \geq N$ and focus on user 1 and user 2. Since we are considering outerbounds, we can assume that

- user 1 can decode W_{12}, W_{13} based on Y_1^n and the side information W_{21}, W_{31};
- user 2 can decode W_{21}, W_{23} based on Y_2^n and the side information W_{12}, W_{32}.

Since user 1 can decode W_{12}, if we provide it with the message W_{32} as genie information, it has the same side information as user 2. Note that the received signals at user 1 and user 2 are

$$\mathbf{Y}_1 = \mathbf{H}_{1R}\mathbf{X}_R + \mathbf{Z}_1; \tag{21}$$

$$\mathbf{Y}_2 = \mathbf{H}_{2R}\mathbf{X}_R + \mathbf{Z}_2. \tag{22}$$

Using the same technique as deriving outerbound for the MIMO interference channel in [12], we perform singular value decomposition to obtain $\mathbf{H}_{2R} = \mathbf{U}\mathbf{\Sigma}_{2R}\mathbf{V}^\dagger$, and we have

$$\mathbf{U}^\dagger\mathbf{Y}_2 = \mathbf{\Sigma}_{2R}\mathbf{V}^\dagger\mathbf{X}_R + \mathbf{U}^\dagger\mathbf{Z}_2 \tag{23}$$

With some change of notation, we have

$$\mathbf{Y}_2' = \mathbf{\Sigma}_{2R}\mathbf{X}_R' + \mathbf{Z}_2', \tag{24}$$

where $\mathbf{Z}_2'$ has the same distribution as $\mathbf{Z}_2$, and $\mathbf{Y}_2' = \mathbf{U}^\dagger\mathbf{Y}_2$, $\mathbf{X}_R' = \mathbf{V}^\dagger\mathbf{X}_R$, $\mathbf{Z}_2' = \mathbf{U}^\dagger\mathbf{Z}_2$. Note that the above operation is invertible and does not change the capacity region. We further consider an enhanced version of the received signal at user 1:

$$\mathbf{Y}_1^{enhance} = \mathbf{H}_{1R}\mathbf{X}_R + \mathbf{N}_1 \tag{25}$$

where $\mathbf{N}_1 \sim \mathcal{CN}(0, \mathbf{K})$

$$\mathbf{K} = \mathbf{I}_M - \mathbf{H}_{1R}\left(\mathbf{H}_{1R}^\dagger\mathbf{H}_{1R}\right)^{-1}\mathbf{H}_{1R}^\dagger + \frac{1}{\alpha}\mathbf{H}_{1R}\mathbf{H}_{1R}^\dagger \tag{26}$$

and α is the maximum among the singular values of matrix $\mathbf{H}_{1R}$ and $\mathbf{H}_{2R}$. Note that using the enhanced version of $\mathbf{Y}_1$ will not reduce the capacity region. Since we have $M \geq N$, we let user 1 to perform zero-forcing to obtain

$$\mathbf{Y}_1' = \mathbf{X}_R' + \mathbf{N}_1' \tag{27}$$

where

$$\mathbf{N}_1' = \mathbf{V}^\dagger\left(\mathbf{H}_{1R}^\dagger\mathbf{H}_{1R}\right)^{-1}\mathbf{H}_{1R}^\dagger\mathbf{N}_1. \tag{28}$$

It is easy to see that $\mathbf{N}_1' \sim \mathcal{N}(0, \frac{1}{\alpha}\mathbf{I}_M)$. Therefore $\mathbf{Y}_1'$ is less noisy than $\mathbf{Y}_2'$. With the genie information W_{32}, we can decode W_{12}, W_{13}, W_{23} using $\mathbf{Y}_1'$ and side information W_{21}, W_{31}. Similarly, we can apply the same argument for user 4 and user 5, and we can decode W_{45}, W_{46}, W_{56} using $\mathbf{Y}_4'$, which is an enhanced version of $\mathbf{Y}_4$, with side information W_{54}, W_{64} and genie information W_{65}. Now we have

$$\begin{aligned}
&nR_{12} + nR_{13} + nR_{23} + nR_{45} + nR_{46} + nR_{56} && (29)\\
&= H(W_{12}W_{13}W_{23}W_{45}W_{46}W_{56}|W_{32}W_{21}W_{31}W_{65}W_{54}W_{64}) \\
&\leq I(W_{12}W_{13}W_{23}W_{45}W_{46}W_{56}; \mathbf{Y}_1'^n\mathbf{Y}_4'^n \\
&\qquad |W_{32}W_{21}W_{31}W_{65}W_{54}W_{64}) + n\epsilon_n && (30)\\
&\leq h(\mathbf{Y}_1'^n\mathbf{Y}_4'^n) - h(\mathbf{Y}_1'^n\mathbf{Y}_4'^n|\mathbf{X}_R^n) && (31)\\
&= I(\mathbf{X}_R^n; \mathbf{Y}_1'^n\mathbf{Y}_4'^n) && (32)
\end{aligned}$$

where equation (31) is because $\mathbf{Y}_1'^n, \mathbf{Y}_4'^n$ are conditionally independent of the messages given $\mathbf{X}_R^n$. From equation (32), we can obtain the DoF upper bound $\min\{2M, N\} = N$ for the 6 messages. We can apply the same method for the other 6 messages to obtain the DoF upper bound $2N$ for the sum capacity.

When $M < N$, we can add antennas at the users such that all the users and the relay have N antennas, which yields the same DoF upper bound $2N$.

Using the cut set bound, we can obtain a DoF upper bound $3\min\{2M, N\}$. Together with the DoF upper bound $2N$, we conclude that

$$DoF \leq 2\min\{3M, N\}. \tag{33}$$

■

Remark 2: This DoF upper bound implies that increasing the number of users in each cluster cannot provide a DoF gain beyond $2N$. This means that sharing the signal space of the relay between 2 users is indeed the optimal approach.

B. Achieving the DoF Upper Bound

Theorem 2: The optimal DoF for the two-cluster multi-way relay channel with 3 users in each cluster is

1) $DoF = 2N$, if $N \leq \frac{12}{7}M$;
2) $DoF = 6M$, if $N \geq 6M$.

Proof: Due to limited space, we only provide a outline for the proof.

Case 1: $N \leq M$. For this case, it is sufficient for the users to use only N antennas to transmit. For each transmission, we can only allow two of the users in the same cluster to be active, which reduces the channel to a two-way relay channel. The functional-decode-and-forward scheme can achieve DoF $2N$.

Case 2: $M < N \leq \frac{12}{7}M$. From the results for the 2-user case, any pair of users in one cluster can share a $(2M-N)$-dimensional space at the relay. Since there are 6 pair of messages that can share the signal space at the relay, the total number of dimensions of the signal space that can be shared by users is $6(2M-N)$. To achieve DoF $2N$, we require all the signal space at the relay is shared by the users, and this yields $6(2M-N) \geq N$, which is equivalent as $N \leq \frac{12}{7}M$.

Case 3: $N \geq 6M$. For this case, the DoF $6M$ can simply be achieved by allowing the relay to decode all the messages and then broadcast the messages to the intended users, since it has sufficient spatial dimensions. ∎

V. Conclusion

In this work, we have studied the degrees of freedom (DoF) of the two-cluster multi-way relay channel. For the case with 2 users in each cluster with arbitrary number of antennas. We have derived a DoF upper bound based on cut set bound, and have shown that it is tight for several nontrivial cases. Conditions when the DoF upper bound can be achieved are established based on the relative number of antennas between the users and the relay. For the case with 3 users in each cluster in the symmetric setting, i.e., when all the users have the same number of antennas. We have derived a new DoF upper bound using a genie-aided approach and channel enhancement, and have shown that the bound is tight using signal space alignment for several scenarios of interests. This result for the 3 user case shows that there is no further DoF gain by increasing the number of users in each cluster when the number of antennas at the relay is limited.

Appendix A
Proof of Lemma 1

Proof: We first consider the case when $p \geq q_1 \geq q_2$ and $q_1 + q_2 > p$. Note that equation (11) is equivalent as

$$\begin{bmatrix} \mathbf{I} & \mathbf{H}_1 & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{H}_2 \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ \mathbf{u}_i \\ \mathbf{w}_i \end{bmatrix} = \mathbf{0}. \tag{34}$$

The null space of the matrix

$$\begin{bmatrix} \mathbf{I} & \mathbf{H}_1 & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{H}_2 \end{bmatrix} \tag{35}$$

has dimension $q_1 + q_2 - p$. It is easy to see that if $q_1 + q_2 > p$, then we can find $q_1 + q_2 - p$ non-zero linearly independent vectors of the form

$$\begin{bmatrix} \mathbf{v}_i & \mathbf{u}_i & \mathbf{w}_i \end{bmatrix}^T \tag{36}$$

from the null space of the matrix shown in equation (35). It remains to see whether all these vectors satisfy $\mathbf{v}_i \neq 0$. Since $p \geq q_1 \geq q_2$, we can see that the null space of matrices $\mathbf{H}_1$ and $\mathbf{H}_2$ has dimension 0. Therefore for all the non-zero vectors satisfying equation (34), we must have $\mathbf{v}_i \neq 0$.

Similarly, when $q_1 \geq p \geq q_2$, we can find $q_1 + q_2 - p$ non-zero linearly independent vectors of the form shown in equation (36) to satisfy equation (34). However, for this case, if we consider the equation $\mathbf{v}_i = \mathbf{H}_2\mathbf{w}_i$, we can see that there are at most q_2 non-zero linearly independent vectors $\mathbf{v}_i$ satisfying this equation. In fact, since $q_1 \geq p$, the null space of matrix $\mathbf{H}_1$ has dimension $q_1 - p$. When we set $\mathbf{w}_i$ and $\mathbf{v}_i$ to 0, we can find $q_1 - p$ non-zero linearly independent vectors $\mathbf{u}_i$ to satisfy equation (34). Therefore we can conclude that among all vectors of the form in equation (36) satisfying equation (34), we can only find q_2 non-zero linearly independent vectors $\mathbf{v}_i$. ∎

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