Signal Space Alignment and Degrees of Freedom for the Two-Cluster Multi-way Relay Channel

(Invited Paper)

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***Abstract*—This paper investigates the degrees of freedom (DoF) of the multi-way relay channel. Specifically, the focus is on the case when there are two clusters and each cluster has two users that wish to exchange messages within the cluster with the help of the relay. It is assumed that the relay has N antennas and each of the users has M antennas. A DoF outerbound is derived and it is shown that for several scenarios of interests the DoF outerbound can be achieved. Specifically, when $N \leq M$, a network coding based two-way relaying approach with time division multiplex access (TDMA) is sufficient to achieve the DoF outerbound. When $M < N \leq \frac{4}{3}M$, the signal space alignment with multiple access and broadcast transmission between the clusters can achieve the DoF outerbound. It is also show that when $N \geq 4M$, the DoF outerbound can be achieved by just using multiple access and broadcast transmission between the users.**

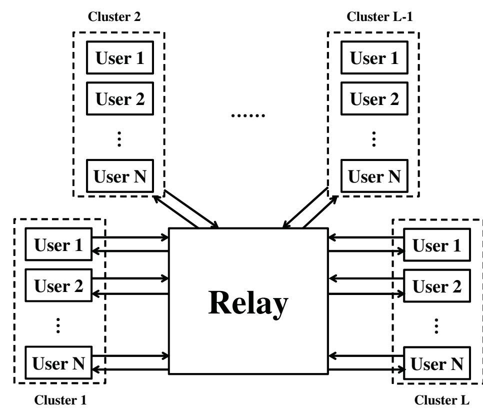

Fig. 1. The multi-way relay channel.

I. Introduction

The multi-way relay channel [1] is a fundamental building block for the relay networks with multicast transmission. Fig. 1 shows a general multi-way relay channel that consists of a relay and L clusters of users with N users in each cluster. The users in each cluster are interested in exchanging information from other users in the same cluster. It is assumed that the users cannot directly communicate with each other, and thus the users in each cluster can only exchange information with the help of the relay. This fundamental model characterizes many interesting practical communication scenarios. For example, in ad hoc networks, wireless nodes can be geographically separated, yet they can communicate to a central controller to share information. This model can also characterize a social network, where users want to exchange information within only a specific set of users using a cellular network, and the base station can serve as the relay. This model is also useful to characterize satellite communications, where the satellite serves as the relay and the users have multicast information that needs to be shared with the help of the satellite.

The two-way relay channel, which is a special case for the general multi-way relay channel, has been extensively studied in [2]–[6] and the references therein. In two-way relay channels, the physical layer network coding is essential to improve the achievable rates. Using network coding, the users can utilize their own message as side information to decode the intended signal. The resources of the relay, under this approach, can be utilized simultaneously for multiple users to maximize the achievable rate.

The multi-way relay channel, on the other hand, contains an arbitrary number of groups of users that want to share information, and the relay needs to handle interference caused by simultaneous transmission from different clusters. The strategies designed for the two-way relay channel are no longer sufficient and more sophisticated strategies need to be designed. Reference [1] has characterized the outerbounds on capacity region and established achievable rates based on decode-and-forward (DF), compress-and-forward (CF), amplify-and-forward (AF), and nested lattice codes. Reference [7] has considered the special case when there is one cluster with full data exchange, and capacity region is derived under the finite field assumption. It is shown that for this case, functional-decode-forward (FDF) combined with rate splitting and joint source-channel decoding achieves capacity. References [8], [9] have investigated the Y channel, which is another special case of the multi-way relay channel with 3 users that want to exchange information. The linear deterministic model is first considered in [8], where the authors have derived a new upper bound and established capacity region. The result is then generalized to the Gaussian channel in [9], where a constant gap result is derived. Reference [10] has studied the case when there are multiple pairs of users with single antenna that need to exchange information with each other, and established a constant gap capacity result. For the general multi-way relay channel, the capacity remains unknown.

The above works mainly focus on the exact capacity characterization of the multi-way relay channel. However, due to the complexity of the general multi-way relay channel, exact characterization of capacity region is difficult, in turn making it difficult to obtain design insights for the general setting. In this work, we will instead focus on another important metric for wireless systems, which is the degrees of freedom (DoF).

The DoF, which is also known as the multiplexing gain, characterizes the high signal-to-noise ratio (SNR) performance of wireless networks. Interference alignment [11] has been shown to achieve the optimal DoF for various wireless multi-terminal network models [11]–[14]. The essence of interference alignment is to keep the interference signals in the smallest number of time/frequency/space dimensions, and therefore the number of independent data streams can be maximized to achieve the optimal DoF. A similar concept, signal space alignment, is proposed in reference [15] for the Y channel. The authors have shown that, by aligning the signals from the users that want to exchange information at the same dimensions, network coding can be utilized to maximize the utilization of the spatial dimension available at the relay to achieve the optimal DoF. In comparison, the goal of signal space alignment is to align the useful signals together to maximize the utilization of the relay, whereas the goal of interference alignment is to align the harmful signals together to minimize the effect of interference. The signal space alignment idea is then extended to the K-user Y channel in reference [16], where achievable DoF is established. The signal space alignment is further utilized in reference [17], which considered a special case of the multi-way relay channel where a base station wants to exchange information with K users with the help of a relay. The DoF of this model is established under some specific relation between the number of antennas at the relay and the users.

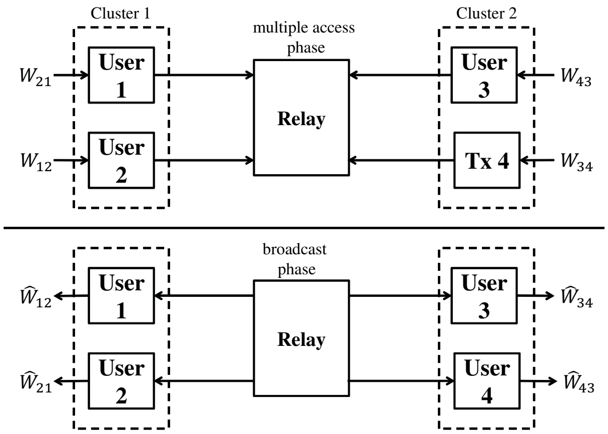

Fig. 2. The multi-way relay channel with two clusters.

In this paper, we investigate the DoF of the multi-way relay channel in order to obtain design insights for optimal transmission strategies. As a first step, we focus on the case when there are two clusters of users and each cluster has two users with multiple antennas at each node. It is assumed that each user has M antennas and the relay has N antennas. Each node is assumed to have global instant channel state information (CSI). We first derive DoF outerbounds based on the cut set bound. We then establish DoF optimal strategies for several scenarios of interests. Specifically, when $N \leq M$, we show that a network coding based two-way relaying approach with time division multiplex access (TDMA) achieves the DoF outerbound. For the case when $M < N \leq \frac{4}{3}M$, we show that the signal space alignment approach achieves the DoF outerbound. An interesting observation for this case is that to achieve the optimal DoF, the users do not need to use all the antennas to exchange information. For the case when $4M \leq N$, we show that a multiple access transmission followed by a broadcast transmission is DoF optimal.

II. System Model

Fig. 2 shows the multi-way relay channel with two clusters, each having two users. In the first cluster, user 1 has a message W_{21} for user 2, and user 2 has a message W_{12} for user 1. In the second cluster user 3 has a message W_{43} for user 4, and user 4 has a message W_{34} for user 3. There are no direct links between any pairs of users, and thus the users can only communicate via the relay [1].

We assume that the relay has N antennas and each of the users has M antennas. All nodes in this network are assumed to be full-duplex. The transmission has two phases, i.e., the multiple access (MAC) phase and broadcast (BC) phase. Note that since all the nodes are assumed to be full-duplex, the MAC phase and BC phase, in reality, appear simultaneously during the transmission.

MAC phase: The transmitted signal from user i for slot t is denoted as $\mathbf{X}_i(t) \in \mathbb{C}^M$, where $i = 1, 2, 3, 4$. The received signal at the relay for slot t is denoted as $\mathbf{Y}_R(t) \in \mathbb{C}^N$, where

$$\mathbf{Y}_R(t) = \sum_{i=1}^{4} \mathbf{H}_{Ri}\mathbf{X}_i(t) + \mathbf{Z}_R(t). \tag{1}$$

BC phase: At slot t, the transmitted signal $\mathbf{X}_R(t) \in \mathbb{C}^N$ from the relay is a function of its received signal from slot 1 to slot $t-1$, i.e.,

$$\mathbf{X}_R(t) = f(\mathbf{Y}_R(1), \cdots, \mathbf{Y}_R(t-1)). \tag{2}$$

The received signal at the users for slot t is then denoted as $\mathbf{Y}_i(t) \in \mathbb{C}^M$ with

$$\mathbf{Y}_i(t) = \mathbf{H}_{iR}\mathbf{X}_R(t) + \mathbf{Z}_i(t). \tag{3}$$

In the above expressions, $\mathbf{H}_{iR} \in \mathbb{C}^{M\times N}$ and $\mathbf{H}_{Ri} \in \mathbb{C}^{N\times M}$ are the channel matrices from the relay to the users and the channel matrices from the users to the relay, respectively. It is also assumed that each entry of the channel matrices are drawn independently from a continuous distribution, which guarantees that each of the channel matrices is full rank almost surely. $\mathbf{Z}_i(t) \in \mathbb{C}^M, \mathbf{Z}_R(t) \in \mathbb{C}^N$ are additive white Gaussian noise vectors with identity covariance matrices. The transmitted signals from the users and the relay satisfy the following power constraint:

$$E\left[\mathrm{tr}\left(\mathbf{X}_i(t)\mathbf{X}_i(t)^\dagger\right)\right] \le P \quad E\left[\mathrm{tr}\left(\mathbf{X}_R(t)\mathbf{X}_R(t)^\dagger\right)\right] \le P \tag{4}$$

We assume the rate of message W_{ij} is $R_{ij}(P)$ for power constraint P. We define $\mathcal{C}(P)$ as the set of all achievable rate tuples $\{R_{ij}(P)\}$ for power constraint P. The degrees of freedom is defined as

$$DoF = \lim_{P\to\infty} \frac{R_{\sum}(P)}{\log(P)}, \tag{5}$$

where $R_{\sum}(P)$ is the sum capacity under power constraint P.

III. Outerbounds on the Degrees of Freedom

In this section, we derive outerbounds on the DoF of the two-cluster multi-way relay channel with two users in each cluster.

Proposition 1: For the multi-way relay channel with two clusters of users and two users in each cluster, when each user is equipped with M antennas and the relay is equipped with N antennas, the DoF outerbound can be characterized as follows:

$$DoF \le 2N \text{ if } N \le 2M \tag{6}$$

$$DoF \le 4M \text{ if } N > 2M. \tag{7}$$

Proof: The outerbound can be established using cut-set bounds. We first combine user 1 and user 3, and also combine user 2 and user 4, which results in a two-way relay channel with two users each having $2M$ antennas and one relay having N antennas. We can then upper bound the rates as follows:

$$R_{21}+R_{43} \le \min\{I(\mathbf{X}_1\mathbf{X}_3;\mathbf{Y}_R), I(\mathbf{X}_R;\mathbf{Y}_2\mathbf{Y}_4)\} \tag{8}$$

$$R_{12}+R_{34} \le \min\{I(\mathbf{X}_2\mathbf{X}_4;\mathbf{Y}_R), I(\mathbf{X}_R;\mathbf{Y}_1\mathbf{Y}_3)\}. \tag{9}$$

If we denote the DoF for message W_{ij} as d_{ij}, then it is easy to see that

$$d_{21}+d_{43} \le \min\{2M, N\}, \tag{10}$$

$$d_{12}+d_{34} \le \min\{2M, N\}. \tag{11}$$

The sum DoF is then upper bounded by

$$DoF = d_{12}+d_{21}+d_{34}+d_{43} \le 2\min\{2M, N\}. \tag{12}$$

The proposition follows directly from equation (12). ∎

IV. Achieving the Degrees of Freedom Outerbounds

In this section, we discuss several cases of the multi-way relay channel described in Section II when the DoF outerbounds can be achieved, which provides us with insights on how to design transmission strategies that are DoF optimal.

Theorem 1: For the multi-way relay channel described in Section II, the optimal DoF is

$$DoF = 2N, \qquad N \le M \tag{13}$$

$$DoF = 2N, \qquad M < N \leq \frac{4}{3}M \tag{14}$$

$$DoF = 4M, \qquad N \geq 4M \tag{15}$$

In the next few sections, we describe the transmission strategies that can achieve the optimal DoF for the scenarios described in *Theorem 1*.

A. $M \geq N$

For this case, the users have more antennas than the relay does. The DoF outerbounds we derived in Section III show that we cannot achieve DoF greater than $2N$. It is easy to see that a network coding based two-way relaying approach with TDMA between the clusters can achieve DoF $2N$. To see this, we only let user 1 and user 2 transmit and let user 3 and user 4 remain silent. In addition, we only allow user 1 and 2 to use N out of the M antennas to transmit. Suppose $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N$ are N linearly independent vectors that span $\mathbb{C}^N$. User 1 sends N independent data streams $d_{21,i}, i = 1, \cdots, N$ along directions $\mathbf{H}_{R1}^{-1}\mathbf{e}_i$, where $\mathbf{H}_{R1} \in \mathbb{C}^{N\times N}$. Similarly, user 2 sends N independent data streams $d_{12,i}, i = 1, \cdots, N$ along directions $\mathbf{H}_{R2}^{-1}\mathbf{e}_i$. Therefore the received signal at the relay is

$$\mathbf{Y}_R = \mathbf{H}_{R1}\sum_{i=1}^{N}\mathbf{H}_{R1}^{-1}\mathbf{e}_i d_{21,i} + \mathbf{H}_{R2}\sum_{i=1}^{N}\mathbf{H}_{R2}^{-1}\mathbf{e}_i d_{12,i} \tag{16}$$

$$= \sum_{i=1}^{N}\mathbf{e}_i\left(d_{21,i} + d_{12,i}\right), \tag{17}$$

where the noise terms are removed from the expressions for clarity since we are considering the high SNR performance.

The relay can then decode the sums of the data streams $d_{21,i} + d_{12,i}$ and broadcast them to the users. Since user 1 and user 2 have $M \geq N$ antennas, they can separate the N network coded data streams to decode the intended messages. Therefore the DoF $2N$ can be achieved.

In the above scheme, we only allow user 1 and user 2 to transmit. Using a TDMA approach, we can easily incorporate user 3 and user 4 to exchange their messages via the relay, to achieve the sum DoF $2N$.

B. $M < N \leq 2M$

For this case, the DoF outerbounds we derived in Section III indicate that we cannot achieve more than $2N$ DoF. Using a scheme similar to the one we used in Section IV-A, we can achieve $2M$ DoF, which is strictly less than the DoF outerbound $2N$. Therefore the network coding based two-way relaying approach with TDMA between the clusters is no longer optimal. Under this scenario, we show that, if $M < N \leq \frac{4}{3}M$, the DoF $2N$ can be achieved using the signal space alignment [15].

1) MAC phase: To show the achievability of DoF $2N$, we let each user send $\frac{1}{2}N$ independent data streams using $\frac{3}{4}N$ antennas. The transmitted signals from user 1 and user 2 are

$$\mathbf{X}_1 = \sum_{i=1}^{N/2}\mathbf{v}_{1,i}d_{21,i} \tag{18}$$

$$\mathbf{X}_2 = \sum_{i=1}^{N/2}\mathbf{v}_{2,i}d_{12,i}, \tag{19}$$

where the beamforming vectors $\mathbf{v}_{k,i}$ are to be determined later.

Similarly, the transmitted signals from user 3 and user 4 are

$$\mathbf{X}_3 = \sum_{i=1}^{N/2}\mathbf{v}_{3,i}d_{43,i} \tag{20}$$

$$\mathbf{X}_4 = \sum_{i=1}^{N/2}\mathbf{v}_{4,i}d_{34,i}. \tag{21}$$

The received signal at the relay is

$$\mathbf{Y}_R(t) = \sum_{i=1}^{4}\mathbf{H}_{Ri}\mathbf{X}_i(t). \tag{22}$$

In order to achieve the DoF $2N$, all of the users must be able to recover the $\frac{N}{2}$ data streams that contains the intended messages. Since the relay only have N antennas, we need to use the signal space alignment concept proposed in reference [15], which aligns the data streams that contain the mutually intended messages. Specifically, we wish to align each pair of the data streams $d_{12,i}$ and $d_{21,i}$, as well as each pair of the data streams $d_{34,i}$ and $d_{43,i}$. To this end, we require vectors $\mathbf{v}_{1,i}$ and $\mathbf{v}_{2,i}$ to satisfy

$$\begin{bmatrix} \mathbf{I} & \mathbf{H}_{R1} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{H}_{R2} \end{bmatrix}\begin{bmatrix} \mathbf{q}_{1,i} \\ \mathbf{v}_{1,i} \\ \mathbf{v}_{2,i} \end{bmatrix} = \mathbf{0}. \tag{23}$$

It is easy to see that we can find exactly $\frac{N}{2}$ pairs of such vectors $\mathbf{v}_{1,i}$ and $\mathbf{v}_{2,i}$ from the null space of the matrix

$$\begin{bmatrix} \mathbf{I} & \mathbf{H}_{R1} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{H}_{R2} \end{bmatrix}. \tag{24}$$

Similarly, we can find exactly $\frac{N}{2}$ pairs of vectors $\mathbf{v}_{3,i}$ and $\mathbf{v}_{4,i}$ in a similar fashion such that

$$\mathbf{q}_{2,i} = \mathbf{H}_{R3}\mathbf{v}_{3,i} = \mathbf{H}_{R4}\mathbf{v}_{4,i} \tag{25}$$

with $\mathbf{q}_{2,i}$ linearly independent for each $i = 1, \cdots, \frac{N}{2}$.

We can now rewrite equation (22) as

$$\begin{aligned} \mathbf{Y}_R &= \sum_{i=1}^{N/2} \mathbf{H}_{R1}\mathbf{v}_{1,i}d_{21,i} + \sum_{i=1}^{N/2} \mathbf{H}_{R2}\mathbf{v}_{2,i}d_{12,i} \\ &+ \sum_{i=1}^{N/2} \mathbf{H}_{R3}\mathbf{v}_{3,i}d_{43,i} + \sum_{i=1}^{N/2} \mathbf{H}_{R4}\mathbf{v}_{4,i}d_{34,i} \qquad (26) \\ &= \sum_{i=1}^{N/2} \mathbf{q}_{1,i}\left(d_{12,i} + d_{21,i}\right) + \sum_{i=1}^{N/2} \mathbf{q}_{2,i}\left(d_{34,i} + d_{43,i}\right) \qquad (27) \end{aligned}$$

Denote $\mathbf{Q}_1$ as the matrix whose columns are composed of vectors $\mathbf{q}_{1,i}$, and $\mathbf{Q}_2$ as the matrix whose columns are composed of vectors $\mathbf{q}_{2,i}$. It remains to see whether $[\mathbf{Q}_1\ \mathbf{Q}_2]$ have full rank. Suppose $[\mathbf{Q}_1\ \mathbf{Q}_2]$ is rank defective, then there exist a vector $\mathbf{v}$ such that

$$\mathbf{v} = \mathbf{H}_{R1}\mathbf{v}_{1,i} = \mathbf{H}_{R2}\mathbf{v}_{2,i} = \mathbf{H}_{R3}\mathbf{v}_{3,i} = \mathbf{H}_{R4}\mathbf{v}_{4,i}. \qquad (28)$$

This is equivalent as

$$\begin{bmatrix} \mathbf{I} & \mathbf{H}_{R1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{H}_{R2} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{H}_{R3} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_{R4} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{v}_{1,i} \\ \mathbf{v}_{2,i} \\ \mathbf{v}_{3,i} \\ \mathbf{v}_{4,i} \end{bmatrix} = \mathbf{0} \qquad (29)$$

Since the dimension of the matrix on the left-hand side is $4N \times 4N$, with probability one, the data streams transmitted along the vectors $\mathbf{v}_{1,i}, \mathbf{v}_{2,i}$ will occupy different signal dimensions with the data streams transmitted along the vectors $\mathbf{v}_{3,i}, \mathbf{v}_{4,i}$ when arriving at the relay. Therefore the relay can use zero-forcing to separate the network coded data streams $d_{12,i}+d_{21,i}$ and $d_{34,i}+d_{43,i}$ for $i = 1, \cdots, \frac{N}{2}$.

2) Broadcast phase: After decoding the network coded data streams, the relay now needs to send $c_{1,i} = d_{12,i} + d_{21,i}$ to user 1 and user 2, and in the meantime it needs to send $c_{2,i} = d_{34,i} + d_{43,i}$ to user 3 and user 4, which can allow the users to decode the intended messages. To this end, we let the relay transmit the following signal to the users:

$$\mathbf{X}_R = \sum_{i=1}^{N/2} \mathbf{u}_{1,i}c_{1,i} + \sum_{i=1}^{N/2} \mathbf{u}_{2,i}c_{2,i} \qquad (30)$$

where $\mathbf{u}_{k,i} \in \mathbb{C}^N, k = 1, 2$ is the beamforming vector which is to be determined later.

When the user k receives the signal transmitted from the relay, it employs a receiver side filter $\mathbf{W}_k \in \mathbb{C}^{\frac{N}{2} \times \frac{3N}{4}}$ to decode the intended signals. The matrices $\mathbf{W}_k, k = 1, 2$ will be jointly designed with the beamforming vectors $\mathbf{u}_{k,i}$ at the relay to guarantee that the unintended signals are zero forced, i.e., we need

$$\mathbf{W}_1\mathbf{H}_{1R}\mathbf{u}_{2,j} = 0 \quad \mathbf{W}_2\mathbf{H}_{2R}\mathbf{u}_{2,j} = 0 \qquad (31)$$

$$\mathbf{W}_3\mathbf{H}_{3R}\mathbf{u}_{1,j} = 0 \quad \mathbf{W}_4\mathbf{H}_{4R}\mathbf{u}_{1,j} = 0. \qquad (32)$$

If we can find the matrices $\mathbf{W}_k$ such that

$$\operatorname{span}\left(\mathbf{W}_1\mathbf{H}_{1R}\right) = \operatorname{span}\left(\mathbf{W}_2\mathbf{H}_{2R}\right) \qquad (33)$$

$$\operatorname{span}\left(\mathbf{W}_3\mathbf{H}_{3R}\right) = \operatorname{span}\left(\mathbf{W}_4\mathbf{H}_{4R}\right), \qquad (34)$$

then the dimension of the product matrix $\mathbf{W}_k\mathbf{H}_{kR}$ guarantees the existence of $\frac{N}{2}$ beamforming vectors $\mathbf{u}_{k,i}$ for $k = 1, 2$ at the relay satisfying equations (31) and (32).

Similar to equation (23), if we can find vectors $\mathbf{w}_i, \mathbf{w}_{1,i}, \mathbf{w}_{2,i}$ such that

$$\begin{bmatrix} \mathbf{w}_i^T & \mathbf{w}_{1,i}^T & \mathbf{w}_{2,i}^T \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{H}_{1R} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{2R} \end{bmatrix} = \mathbf{0}, \qquad (35)$$

then the dimension of the block matrix on the left-hand side is $\frac{5}{2}N \times 2N$, which guarantees that there exist $\frac{N}{2}$ linearly independent vectors

$$\begin{bmatrix} \mathbf{w}_i^T & \mathbf{w}_{1,i}^T & \mathbf{w}_{2,i}^T \end{bmatrix}. \qquad (36)$$

The vectors $\mathbf{w}_{1,i}^T$ and $\mathbf{w}_{2,i}^T$ can be used as rows to generate the receiver side filter matrices $\mathbf{W}_1$ and $\mathbf{W}_2$. Since the channel coefficients are generated according to a continuous distribution, the matrix on the left-hand side of equation (35) has full rank almost surely. Thus we can find $\frac{N}{2}$ vectors (36) from its left null space. The beamforming vectors $\mathbf{u}_{2,i}$ at the relay can then be found from the right null space of $\mathbf{W}_1\mathbf{H}_{1R}$. Note that with the chosen matrices $\mathbf{W}_1$ and $\mathbf{W}_2$, we have $\mathbf{W}_1\mathbf{H}_{1R} = \mathbf{W}_2\mathbf{H}_{2R}$, which is a stronger condition than (33). We can generate the matrices $\mathbf{W}_3$ and $\mathbf{W}_4$, and the beamforming vectors $\mathbf{u}_{1,i}$ in a similar fashion. Using similar method as in equation (29), we can guarantee that with probability 1, there is no intersection between the column space of $\mathbf{W}_1\mathbf{H}_{1R}$ and $\mathbf{W}_3\mathbf{H}_{3R}$.

Following from the fact that the channel coefficients are drawn from a continuous distribution, we can guarantee that the vectors $\mathbf{W}_1\mathbf{H}_{1R}\mathbf{u}_{1,i}$ at user 1 are linearly independent almost surely for $i = 1, \cdots, \frac{N}{2}$. Similar arguments hold for user 2, user 3 and user 4. Therefore user 1 and user 2 can decode the network coded data streams $c_{1,i} = d_{12,i} + d_{21,i}$ and user 3 and user 4 can decode the network coded data streams $c_{2,i} = d_{34,i} + d_{43,i}$, and consequently, the intended messages can be decoded. In conclusion, with the signal space alignment approach, the users can exchange $2N$ independent data streams and the DoF $2N$ can be achieved.

C. $N \geq 4M$

For this case, the number of antennas at the relay is greater than the total number of antennas of the users. It is easy to see that we can let each user transmit M independent data streams using M antennas, and the relay is able to decode all $4M$ data streams in the MAC phase. In the broadcast phase, the relay can broadcast all the $4M$ data streams to the intended users, and therefore DoF $4M$ can be achieved. Note that for this case the relay has sufficient number of spatial dimensions and it can distinguish all the incoming data streams.

D. Discussion on other cases

For the cases when $\frac{4}{3}M < N \leq 2M$, the DoF outerbounds indicate that we cannot achieve more than $2N$ DoF. However, whether this DoF outerbound is achievable remains unknown at this point. For this case, the signal space alignment approach used in previous sections cannot achieve the optimal DoF. The reason is that the dimension of the intersection of the signal space between the users in one cluster at the relay is less than $\frac{N}{2}$. The users in one cluster can at most utilize $2M - N$ dimensions of the signal space at the relay in common. Therefore, the DoF achieved using the signal space alignment approach is strictly less than $2N$.

For the case when $2M < N < 4M$, the DoF outerbounds indicate that we cannot achieve more than $4M$ DoF. In this scenario, there is no intersection between the signal spaces of the users in the same cluster at the relay, which prohibits the signal space alignment approach. The optimal DoF and the optimal transmission scheme remain unknown and are left as future work.

V. Conclusion

In this paper, we have investigated the degrees of freedom (DoF) of the multi-way relay channel with two clusters and each cluster has two users that wish to exchange messages within the cluster with the help of the relay. It is assumed that the relay has N antennas and each of the users has M antennas. We have derived a DoF outerbound based on the cut set bound. We have shown that when $N \leq M$, a network coding based two-way relaying approach with time division multiplex access (TDMA) between the clusters is sufficient to achieve the DoF outerbound. When $M < N \leq \frac{4}{3}M$, the optimal DoF can be achieved using signal space alignment with multiple access transmission and broadcast transmission between the two clusters. For this case, the users do not need to use all the antennas for transmission. We have also shown that when $N \geq 4M$, using only multiples access transmission and broadcast transmission is sufficient to achieve the DoF outerbounds.

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