Two-way Lossy Compression via a Relay with Self Source

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***Abstract*—We consider interactive source coding of two sources through a relay which also has a source. Alice and Bob have no direct links and wish to exchange their sources with fidelity via an intermediary, Ryan. Ryan also has an individual source and seeks to communicate it to Alice and Bob. We develop inner and outer bounds for the optimal rate-distortion region of this problem, which coincide in certain lossless cases, e.g., when the sources of Alice and Bob are conditionally independent given the source of Ryan or when two of the sources are functions of the third one. The bounds heavily make use of Wyner-Ziv and Berger-Tung coding and often rely on linear network coding. Our results highlight the dual role of the relaying source, which, on one hand, facilitates compression rate savings for the other two sources by helping as side information, and on the other hand, requires additional rate for its own description.**

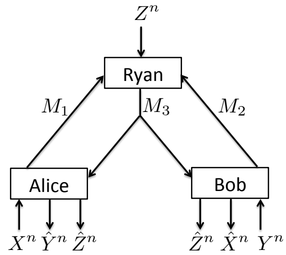

Fig. 1. Two-way lossy compression via a relay with self source

I. Introduction

Interactive lossy compression of two sources is first studied in [1], and is generalized to the case with helper in [2]. Interactive compression of sources involves a relay node when the sources are not linked directly, but through a relay. Interactive lossless compression of two or more sources through a relay is considered and fully characterized in [3]. In this setup, the relay, termed the processing broadcast satellite therein, receives information from all terminals and broadcasts a single information stream back to the terminals who seek to reconstruct all the other sources. The relay has no side information to use or any individual source to compress. The lossy version of the same problem has been investigated in [4] and several inner and outer bounds have been developed, which coincide in certain cases. Other related work include [5] that considers a similar setup but with a side-information-aided relay that broadcasts different streams to different users, [6] that considers a three-terminal interactive setting in which each user either remains silent or broadcasts its source to both of the other users, and [7] that considers a direct interaction between the terminals and the relay merely serves as an intermediate hop in one of the links between the terminals.

There are two critical elements to the analysis of relay-assisted interactive source coding problems. One is the use of simultaneous binning codes [3], [4], [8] and linear network coding [9] to facilitate a single broadcast stream for conveying different information contents to different users. The other element is source compression with decoder-only side information, whose lossless and lossy versions are established in the seminal work by Slepian and Wolf [10] and Wyner and Ziv [11], respectively. The latter is extended to distributed lossy source coding by Berger [12] and Tung [13]. Other closely related problems are distributed lossy source coding with decoder side information [14], lossy compression of a single source for many decoders each with an individual side information [15], cascade source coding [16] with various side information considerations [7], [17]–[20], and lossy broadcast of a two-part source to two terminals each of which have one of the components as side information, a problem termed complementary delivery [21].

In this paper, we consider a generalization of the problems in [3], [4] as illustrated in Figure 1. Alice and Bob wish to exchange *lossy* description of their sources. The communication takes place through a relay, Ryan, that possibly processes their information exchange and broadcasts a single stream back to them. In addition, Ryan has an individual source and wishes to broadcast it to Alice and Bob with a fidelity criterion. We seek to find the set of all compression rates that achieve some given target distortion requirements. We show that Ryan's source can help as side information for compressing the sources of Alice and Bob, but also requires additional rate for its own compression. Depending upon the correlation level, it would be better for Ryan to decode Alice and Bob descriptions before sending his own source, or it might be more reasonable for Ryan to forward Alice and Bob descriptions without decoding. In section II, we formally state the problem formulation. In Sections III and IV, we state and prove several outer and inner bounds for the rate-distortion region of the problem. We conclude the paper in Section V with some remarks and future directions, and leave a technical proof for the Appendix.

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II. System Model

Three correlated memoryless sources (X, Y, Z) take values on alphabets $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$. Alice and Bob have access to X and Y, respectively, and wish to exchange their information through a relay, Ryan, that broadcasts a single stream of information back to Alice and Bob. Ryan has access to Z and wishes to communicate it along with X and Y. Alice is asked to reconstruct lossy descriptions $\hat{Y}$ and $\hat{Z}$, and Bob is required to reconstruct lossy descriptions $\hat{X}$ and $\hat{Z}$, within given distortion levels. No decoding is required at Ryan. This configuration is depicted in Figure 1.

More formally, consider reconstruction alphabets $\hat{\mathcal{X}}$, $\hat{\mathcal{Y}}$, and $\hat{\mathcal{Z}}$, and three additive distortion measures $d_1(x, \hat{x})$, $d_2(y, \hat{y})$, and $d_3(z, \hat{z})$ such that

$$d_1(x^n, \hat{x}^n) := \frac{1}{n} \sum_{t=1}^{n} d_1(x_t, \hat{x}_t),$$

and likewise for the other two measures. Define $[\![i, j]\!]$ to be the set of integers between i and j. An $(n, 2^{nR_1}, 2^{nR_2}, 2^{nR_3})$ code for the problem of interactive source coding via a relay with self source consists of:

- Alice encoder $m_1(x^n) : \mathcal{X}^n \to [\![1, 2^{nR_1}]\!]$ and Alice decoders $\hat{y}^n(m_3, x^n) : [\![1, 2^{nR_3}]\!] \times \mathcal{X}^n \to \hat{\mathcal{Y}}^n$ and $\hat{z}^n_A(m_3, x^n) : [\![1, 2^{nR_3}]\!] \times \mathcal{X}^n \to \hat{\mathcal{Z}}^n$;
- Bob encoder $m_2(y^n) : \mathcal{Y}^n \to [\![1, 2^{nR_2}]\!]$ and Bob decoders $\hat{x}^n(m_3, y^n) : [\![1, 2^{nR_3}]\!] \times \mathcal{Y}^n \to \hat{\mathcal{X}}^n$ and $\hat{z}^n_B(m_3, y^n) : [\![1, 2^{nR_3}]\!] \times \mathcal{Y}^n \to \hat{\mathcal{Z}}^n$;
- Relay encoder $m_3(m_1, m_2, z^n) : [\![1, 2^{nR_1}]\!] \times [\![1, 2^{nR_2}]\!] \times \mathcal{Z}^n \to [\![1, 2^{nR_3}]\!]$.

A rate triple (R_1, R_2, R_3) is called achievable with distortion quadruple $(D_1, D_2, D_{3A}, D_{3B})$ if there exists a sequence of $(n, 2^{nR_1}, 2^{nR_2}, 2^{nR_3})$ codes such that

$$\limsup_{n\to\infty} \mathbb{E}[d_1(X^n, \hat{X}^n)] \leq D_1,$$
$$\limsup_{n\to\infty} \mathbb{E}[d_2(Y^n, \hat{Y}^n)] \leq D_2,$$
$$\limsup_{n\to\infty} \mathbb{E}[d_3(Z^n, \hat{Z}^n_A)] \leq D_{3A},$$
$$\limsup_{n\to\infty} \mathbb{E}[d_3(Z^n, \hat{Z}^n_B)] \leq D_{3B}.$$

The rate-distortion region $\mathcal{R}(D_1, D_2, D_{3A}, D_{3B})$ is the closure of all rate triples (R_1, R_2, R_3) that achieve the distortion quadruple $(D_1, D_2, D_{3A}, D_{3B})$.

Remark 1. Lossless reconstruction corresponds to when all reconstruction alphabets are equal to the corresponding original alphabets, all distortion measures are Hamming distortions, and $D_1 = D_2 = D_{3A} = D_{3B} = 0$.

III. Outer Bounds

Our first outer bound is a cutset bound expressed in terms of the Wyner-Ziv result [11], and the conditional rate-distortion [22] for cooperative lossy source coding [23, p. 306]. In particular, let $R^{\text{WZ}}_{X|YZ}(D)$ denote the rate-distortion function for the source X when the side information pair (Y, Z) is available at the decoder, so that

$$R^{\text{WZ}}_{X|YZ}(D) = \min_{\substack{p(u|x), \hat{x}(u,y,z):\\ \mathbb{E}[d(X,\hat{X})] \leq D}} I(X; U|YZ). \tag{1}$$

In addition, let $R^{\text{SH}}_{XZ|Y}(D_1, D_2)$ denote the rate-distortion function for the super-source (X, Z) when the side information Y is available to both encoder and decoder, so that

$$R^{\text{SH}}_{XZ|Y}(D_1, D_2) = \min_{\substack{p(\hat{x}, \hat{z}|x,z,y):\\ \mathbb{E}[d(X,\hat{X})] \leq D_1, \mathbb{E}[d(Z,\hat{Z})] \leq D_2}} I(XZ; \hat{X}\hat{Z}|Y). \tag{2}$$

The latter result can be proved directly along the lines of standard rate-distortion analysis [23] for shared side information [22] or can be seen as a corollary of the cascade source coding result [16].

We now state our cutset outer bound.

Theorem 1. *If a rate triple (R_1, R_2, R_3) belongs to the rate-distortion region $\mathcal{R}(D_1, D_2, D_{3A}, D_{3B})$, it must satisfy the following conditions:*

$$R_1 \geq R^{WZ}_{X|YZ}(D_1), \tag{3}$$
$$R_2 \geq R^{WZ}_{Y|XZ}(D_2), \tag{4}$$
$$R_3 \geq \max\{R^{SH}_{XZ|Y}(D_1, D_{3B}), R^{SH}_{YZ|X}(D_2, D_{3A})\}. \tag{5}$$

Proof. The bound (3) follows from the cut between the Alice encoder transmitting X and the super-decoder consisting of Bob and Ryan with access to the side information (Y, Z). The bound (4) has a similar nature, with the role of Alice and Bob reversed. The first term in bound (5) follows from a genie-aided cutset bound, in which the super-encoder consisting of Alice and Ryan sends (X, Z) to the Bob decoder whose side information Y is shared with the super-encoder. The second term in bound (5) follows by exchanging the role of Alice and Bob in the previous argument. □

The cutset bound in Theorem 1 is based on rate-distortion functions, in which mutual information terms are individually optimized. One can build upon the techniques in [1], [4] and improve the cutset bound by jointly optimizing the mutual information terms.

Theorem 2. *If a rate triple (R_1, R_2, R_3) belongs to the rate-distortion region $\mathcal{R}(D_1, D_2, D_{3A}, D_{3B})$, it must satisfy the following conditions:*

$$R_1 \geq I(X; U_1|YZ), \tag{6}$$
$$R_2 \geq I(Y; U_2|XZ), \tag{7}$$
$$R_3 \geq \max\{I(XZ; V|YU_2), I(YZ; V|XU_1)\}, \tag{8}$$

for some auxiliary random variables (U_1, U_2, V) that satisfy the Markov chains $U_1 - X - YZ$, $U_2 - Y - XZ$, $V - U_1U_2Z - XY$, $V - XU_2Z - Y$, $V - YU_1Z - X$, and some reconstruction functions $\hat{x}(v, u_2, y)$, $\hat{y}(v, u_1, x)$, $\hat{z}_A(v, u_1, x)$, $\hat{z}_B(v, u_2, y)$ that satisfy $\mathbb{E}[d_1(X, \hat{X}(V, U_2, Y))] \leq D_1$, $\mathbb{E}[d_2(Y, \hat{Y}(V, U_1, X))] \leq D_2$, $\mathbb{E}[d_3(Z, \hat{Z}_A(V, U_1, X))] \leq D_{3A}$, $\mathbb{E}[d_3(Z, \hat{Z}_B(V, U_2, Y))] \leq D_{3B}$.

Proof. See the Appendix. □

Remark 2. For lossless compression, Theorems 1 and 2 reduce to

$$R_1 \geq H(X|YZ), \quad (9)$$
$$R_2 \geq H(Y|XZ), \quad (10)$$
$$R_3 \geq \max\{H(XZ|Y), H(YZ|X)\}. \quad (11)$$

IV. Inner Bounds

In this section, we state four achievability schemes based on the different roles the relay can take. In certain cases, the resulting inner bounds match with the outer bounds of Section III, thus giving a full characterization.

A. Relay decodes, computes, and jointly re-encodes

Theorem 3. *Any rate triple (R_1, R_2, R_3) that satisfies the following conditions belongs to the rate-distortion region $\mathcal{R}(D_1, D_2, D_{3A}, D_{3B})$:*

$$R_1 \geq I(X; U_1|U_2ZQ), \quad (12)$$
$$R_2 \geq I(Y; U_2|U_1ZQ), \quad (13)$$
$$R_1 + R_2 \geq I(XY; U_1U_2|ZQ), \quad (14)$$
$$R_3 \geq \max\{I(WZ; V|YU_2Q), I(WZ; V|XU_1Q)\}, \quad (15)$$

where Q is the time sharing random variable independent of all the other random variables, the auxiliary random variables (U_1, U_2, W, V) satisfy the Markov chains $U_1 - XQ - YZ$, $U_2 - YQ - XZ$, $V - WZQ - XYU_1U_2$, the function $w(x, y)$ satisfies $H(W(X, Y)|U_1, U_2) = 0$, and the reconstruction functions $\hat{x}(v, u_2, y)$, $\hat{y}(v, u_1, x)$, $\hat{z}_A(v, u_1, x)$, $\hat{z}_B(v, u_2, y)$ satisfy $\mathbb{E}[d_1(X, \hat{X}(V, U_2, Y))] \leq D_1$, $\mathbb{E}[d_2(Y, \hat{Y}(V, U_1, X))] \leq D_2$, $\mathbb{E}[d_3(Z, \hat{Z}_A(V, U_1, X))] \leq D_{3A}$, $\mathbb{E}[d_3(Z, \hat{Z}_B(V, U_2, Y))] \leq D_{3B}$.

Proof. Alice and Bob use Berger-Tung coding [12], [13] for distributed lossy compression of (X, Y) as descriptions (U_1, U_2) to Ryan who has access to the side information Z [14]. Ryan then losslessly computes the function $W(X, Y)$ as a consequence of the condition $H(W(X, Y)|U_1, U_2) = 0$, and uses Wyner-Ziv coding for the pair (W, Z) aimed for Alice with side information (X, U_1) and Bob with side information (Y, U_2). Note that the two streams aimed for Alice and Bob need not be sent separately, but can be combined via simultaneous binning codes [3], [4], [8], thus the use of max in the bound (15) for R_3. □

Remark 3. The achievable region in Theorem 3 with the choices $U_1 = X$, $U_2 = Y$, $W = (X, Y)$, and $V = (X, Y, Z)$ reduces for the lossless case to

$$R_1 \geq H(X|YZ), \quad (16)$$
$$R_2 \geq H(Y|XZ), \quad (17)$$
$$R_1 + R_2 \geq H(XY|Z), \quad (18)$$
$$R_3 \geq \max\{H(XZ|Y), H(YZ|X)\}, \quad (19)$$

which is tight if $X - Z - Y$.

B. Relay forwards other sources, then sends his source

Theorem 4. *Any rate triple (R_1, R_2, R_3) that satisfies the following conditions belongs to the rate-distortion region $\mathcal{R}(D_1, D_2, D_{3A}, D_{3B})$:*

$$R_1 \geq I(X; U_1|Y), \quad (20)$$
$$R_2 \geq I(Y; U_2|X), \quad (21)$$
$$R_3 \geq \max\{I(X; U_1|Y), I(Y; U_2|X)\} + \max\{I(Z; V|YU_1U_2), I(Z; V|XU_1U_2)\}, \quad (22)$$

where the auxiliary random variables (U_1, U_2, V) satisfy the Markov chains $U_1 - X - YZ$, $U_2 - Y - XZ$, $V - Z - XYU_1U_2$, and the reconstruction functions $\hat{x}(v, u_2, y)$, $\hat{y}(v, u_1, x)$, $\hat{z}_A(v, u_1, u_2, x)$, $\hat{z}_B(v, u_1, u_2, y)$ satisfy $\mathbb{E}[d_1(X, \hat{X}(V, U_2, Y))] \leq D_1$, $\mathbb{E}[d_2(Y, \hat{Y}(V, U_1, X))] \leq D_2$, $\mathbb{E}[d_3(Z, \hat{Z}_A(V, U_1, U_2, X))] \leq D_{3A}$, $\mathbb{E}[d_3(Z, \hat{Z}_B(V, U_1, U_2, Y))] \leq D_{3B}$.

Proof. Alice and Bob use Wyner-Ziv coding [11] for exchange of (X, Y) with each other via lossy descriptions (U_1, U_2). Ryan merely combines (U_1, U_2) via linear network coding [4], [9] and forwards a linear combination of them without any decoding attempt. The relay then sends his own source Z as a lossy description V aimed for Alice with side information (X, U_1, U_2) and Bob with side information (Y, U_1, U_2). Note that the two streams aimed for Alice and Bob need not be sent separately, but can be again combined via simultaneous binning codes [3], [4], [8]. □

Remark 4. The region in Theorem 4 with the choices $U_1 = X$, $U_2 = Y$, and $V = Z$ reduces for the lossless case to

$$R_1 \geq H(X|Y), \quad (23)$$
$$R_2 \geq H(Y|X), \quad (24)$$
$$R_3 \geq \max\{H(XZ|Y), H(YZ|X)\}, \quad (25)$$

which is tight if Z is independent of the pair (X, Y), i.e., if $I(XY; Z) = 0$.

C. Relay sends his source, then forwards other sources

Theorem 5. *Any rate triple (R_1, R_2, R_3) that satisfies the following conditions belongs to the rate-distortion region $\mathcal{R}(D_1, D_2, D_{3A}, D_{3B})$:*

$$R_1 \geq I(X; U_1|YV), \quad (26)$$
$$R_2 \geq I(Y; U_2|XV), \quad (27)$$
$$R_3 \geq \max\{I(Z; V|Y), I(Z; V|X)\} + \max\{I(X; U_1|YV), I(Y; U_2|XV)\}, \quad (28)$$

where the auxiliary random variables (U_1, U_2, V) satisfy the Markov chains $V - Z - XY$, $U_1 - X - YZV$, $U_2 - Y - XZV$, and the reconstruction functions $\hat{x}(u_1, u_2, v, y)$, $\hat{y}(u_1, u_2, v, x)$, $\hat{z}_A(v, x)$, $\hat{z}_B(v, y)$ satisfy $\mathbb{E}[d_1(X, \hat{X}(V, U_1, U_2, Y))] \leq D_1$, $\mathbb{E}[d_2(Y, \hat{Y}(V, U_1, U_2, X))] \leq D_2$, $\mathbb{E}[d_3(Z, \hat{Z}_A(V, X))] \leq D_{3A}$, $\mathbb{E}[d_3(Z, \hat{Z}_B(V, Y))] \leq D_{3B}$.

Proof. Ryan first sends his own source Z as a lossy description V aimed for Alice with side information X and Bob with side information Y. Note that the two streams aimed for Alice and Bob are combined via simultaneous binning codes [3], [4], [8]. Next, Alice and Bob use Wyner-Ziv coding [11] to exchange (X,Y) with each other via lossy descriptions (U_1,U_2), where now Alice has side information (X,V) and Bob has side information (Y,V). Ryan does not attempt to decode (U_1,U_2), but merely forwards a linear combination of them [4], [9]. □

Remark 5. The region in Theorem 5 with the choices $U_1 = X$, $U_2 = Y$, and $V = Z$ reduces for the lossless case to

$$R_1 \geq H(X|YZ), \tag{29}$$

$$R_2 \geq H(Y|XZ), \tag{30}$$

$$R_3 \geq \max\{H(Z|Y), H(Z|X)\} + \max\{H(X|YZ), H(Y|XZ)\}, \tag{31}$$

which is tight if among the triple (X,Y,Z) two of them are functions of the third one, e.g., $Y = f(X)$ and $Z = g(X)$.

D. Relay decodes part of other sources, jointly re-encodes with his source, then forwards the rest of other sources

Theorem 6. *Any rate triple (R_1,R_2,R_3) that satisfies the following conditions belongs to the rate-distortion region $\mathcal{R}(D_1,D_2,D_{3A},D_{3B})$:*

$$R_1 \geq I(X;U_1|U_2ZQ) + I(X;S_1|VU_2YQ), \tag{32}$$

$$R_2 \geq I(Y;U_2|U_1ZQ) + I(Y;S_2|VU_1XQ), \tag{33}$$

$$R_1 + R_2 \geq I(XY;U_1U_2|ZQ) + I(X;S_1|VU_2YQ) + I(Y;S_2|VU_1XQ), \tag{34}$$

$$R_3 \geq \max\{I(WZ;V|U_2YQ), I(WZ;V|U_1XQ)\} + \max\{I(X;S_1|VU_2YQ), I(Y;S_2|VU_1XQ)\}, \tag{35}$$

where Q is the time sharing random variable independent of (X,Y); the auxiliary random variables (U_1,U_2,W,V,S_1,S_2) satisfy the Markov chains $U_1 - XQ - YZ$, $U_2 - YQ - XZ$, $V - WZQ - XY$, $S_1 - XQ - YZU_1VW$, $S_2 - YQ - XZU_2VW$; the function $w(x,y)$ satisfies $H(W(X,Y)|U_1,U_2) = 0$; and the reconstruction functions $\hat{x}(s_1,v,u_1,y)$, $\hat{y}(s_2,v,u_2,x)$, $\hat{z}_A(v,u_1,x)$, $\hat{z}_B(v,u_2,y)$ satisfy $\mathbb{E}[d_1(X,\hat{X}(S_1,V,U_2,Y))] \leq D_1$, $\mathbb{E}[d_2(Y,\hat{Y}(S_2,V,U_1,X))] \leq D_2$, $\mathbb{E}[d_3(Z,\hat{Z}_A(V,U_1,X))] \leq D_{3A}$, $\mathbb{E}[d_3(Z,\hat{Z}_B(V,U_2,Y))] \leq D_{3B}$.

Proof. Alice and Bob use Berger-Tung coding [12], [13] to send (U_1,U_2) as a first descriptions of (X,Y) to Ryan who has access to the side information Z [14]. Ryan then losslessly computes the function $W(X,Y)$ as a consequence of the condition $H(W(X,Y)|U_1,U_2) = 0$, and uses Wyner-Ziv coding for the pair (W,Z) aimed for Alice with side information (X,U_1) and Bob with side information (Y,U_2). The two streams aimed for Alice and Bob are combined via simultaneous binning codes [3], [4], [8]. Finally, Alice and Bob use Wyner-Ziv coding to exchange a second description of (X,Y) as (S_1,S_2) with each other, where now Alice has side information (X,U_1,V) and Bob has side information (Y,U_2,V). Ryan does not attempt to decode these second descriptions and simply forwards a linear combination of (S_1,S_2) [4], [9]. □

Remark 6. The achievable region in Theorem 6 subsumes Theorems 3 and 5 as special cases. In particular, we can recover Theorem 3 by setting $S_1 = S_2 =$ constant, and Theorem 5 by making the assignments $Q = U_1 = U_2 = W =$ constant, $S_1 = U_1$, and $S_2 = U_2$.

Remark 7. The achievable region in Theorem 6 with the choices $W = (U_1,U_2)$, $S_1 = X$, and $S_2 = Y$ reduces for the lossless case to

$$R_1 \geq I(X;U_1|U_2Z) + H(X|VU_2Y) \tag{36}$$

$$R_2 \geq I(Y;U_2|U_1Z) + H(Y|VU_1X) \tag{37}$$

$$R_1 + R_2 \geq I(XY;U_1U_2|Z) + H(X|VU_2Y) + H(Y|VU_1X) \tag{38}$$

$$R_3 \geq \max\{I(U_1Z;V|U_2Y), I(U_2Z;V|U_1X)\} + \max\{H(X|VU_2Y), H(Y|VU_1X)\}, \tag{39}$$

where the auxiliary random variables (U_1,U_2,V) satisfy the Markov chains $U_1 - X - YZ$, $U_2 - Y - XZ$, $V - U_1U_2Z - XY$.

V. Conclusion

In this paper, we studied a relay-assisted interactive source coding problem, where the relay also has an individual source to communicate. We stated several inner and outer bounds for the rate-distortion region of the problem. Depending upon the relations of the three sources, different strategies can be adopted at the relay. A highly correlated source at the relay is best suited to a scenario in which the relay may partially or fully decode the other two sources and re-compress them along with its own source. However, a weakly correlated source at the relay may merely forward the descriptions of the other two sources and then send its own source, or vice versa. We leave a concrete comparison of these different strategies for the binary and Gaussian sources for future research. An additional future direction is to study the case in which the source at the relay is shared with one of the terminals and is sought to be communicated only to the third terminal.

Appendix
Proof of Theorem 2

We use the converse technique in [1], [4]. Consider any sequence of $(n, 2^{nR_1}, 2^{nR_2}, 2^{nR_3})$ codes that achieves the distortion quadruple (D_1,D_2,D_{3A},D_{3B}). We can lower bound R_1 as follows.

$$\begin{aligned} nR_1 &\geq H(M_1) \geq H(M_1|Y^nZ^n) \geq I(M_1;X^n|Y^nZ^n) \\ &= \sum_{t=1}^{n} [H(X_t|Y_tZ_t) - H(X_t|M_1X^{t-1}Y^nZ^n)] \\ &\geq \sum_{t=1}^{n} I(X_t;U_{1t}|Y_tZ_t), \end{aligned} \tag{40}$$

where (40) follows from conditioning reduces entropy and the definition of $U_{1t} := (M_1 X^{t-1} Y_{t+1}^n Z_{t+1}^n)$. We can similarly bound R_2 as follows.

$$nR_2 \geq \sum_{t=1}^{n} [H(Y_t|X_t Z_t) - H(Y_t|M_2 Y^{t-1} X^n Z^n)]$$
$$\geq \sum_{t=1}^{n} I(Y_t; U_{2t}|X_t Z_t),$$

where $U_{2t} := (M_2 Y^{t-1} X_{t+1}^n Z_{t+1}^n)$. To bound R_3 we write

$$nR_3 \geq H(M_3) \geq H(M_3|M_2 Y^n) \geq I(M_3; X^n Z^n|M_2 Y^n)$$
$$= \sum_{t=1}^{n} [H(X_t Z_t|M_2 Y^n X_{t+1}^n Z_{t+1}^n) - H(X_t Z_t|M_3 M_2 Y^n X_{t+1}^n Z_{t+1}^n)]$$
$$\geq \sum_{t=1}^{n} [H(X_t Z_t|U_{2t} Y_t) - H(X_t Z_t|V_t U_{2t} Y_t)] \quad (41)$$
$$= \sum_{t=1}^{n} I(X_t Z_t; V_t|U_{2t} Y_t),$$

where (41) follows from conditioning reduces entropy, definition of $V_t := M_3$, and the Markov chain $X_t Z_t - U_{2t} Y_t - Y_{t+1}^n$. Analogously, we can write

$$nR_3 \geq \sum_{t=1}^{n} [H(Y_t Z_t|M_1 X^n Y_{t+1}^n Z_{t+1}^n) - H(Y_t Z_t|M_3 M_1 X^n Y_{t+1}^n Z_{t+1}^n)]$$
$$\geq \sum_{t=1}^{n} [H(Y_t Z_t|U_{1t} X_t) - H(X_t Z_t|V_t U_{1t} X_t)]$$
$$= \sum_{t=1}^{n} I(Y_t Z_t; V_t|U_{1t} X_t),$$

due to the Markov chain $Y_t Z_t - U_{1t} X_t - X_{t+1}^n$.

The Markov chains $U_{1t} - X_t - Y_t Z_t$, $U_{2t} - Y_t - X_t Z_t$, $V_t - U_{1t} U_{2t} Z_t - X_t Y_t$, $V_t - X_t U_{2t} Z_t - Y_t$, $V_t - Y_t U_{1t} Z_t - Y_t$ all follow from the definitions of U_{1t}, U_{2t}, and V_t above.

The final step is to prove the expected single-letter distortion measures. The key is to use [23, Lemma 20.2] which states that, if $Y - Z - W$ forms a Markov chain, then for every reconstruction function $\hat{y}(z, w)$, there exists another reconstruction function $\hat{y}^*(z)$ such that

$$\mathbb{E}[d(Y, \hat{y}^*(Z))] \leq \mathbb{E}[d(Y, \hat{y}(Z, W))]. \quad (42)$$

Note for our problem that, for each $t = 1, \cdots, n$, we can define a generalized function $\hat{x}_t'(m_3, y^{t-1}, y_t, y_{t+1}^n, m_2, x_{t+1}^n, z_{t+1}^n) = \hat{x}_t'(v_t, u_{2t}, y_t, y_{t+1}^n)$ equal to the reconstruction function $\hat{x}_t(m_3, y^n)$ for all $m_2, x_{t+1}^n, z_{t+1}^n$. Then, due to the Markov chain $X_t - V_t U_{2t} Y_t - Y_{t+1}^n$, the above lemma would imply that another reconstruction function $\hat{x}_t^*(v_t, u_{2t}, y_t)$ exists such that

$$\mathbb{E}[d_1(X_t, \hat{x}_t^*(V_t, U_{2t}, Y_t))] \leq \mathbb{E}[d_1(X_t, \hat{x}_t(M_3, Y^n))]. \quad (43)$$

Similar arguments can be made about the other three distortion measures. The rest of the proof is standard convexity arguments [23]. This completes the proof of Theorem 2.

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REFERENCES

[1] A. Kaspi, "Two-way source coding with a fidelity criterion," *IEEE Trans. Inf. Theory*, vol. 31, no. 6, pp. 735–740, 1985.

[2] H. Permuter, Y. Steinberg, and T. Weissman, "Two-way source coding with a helper," *IEEE Trans. Inf. Theory*, vol. 56, no. 6, pp. 2905–2919, 2010.

[3] A. Wyner, J. Wolf, and F. Willems, "Communicating via a processing broadcast satellite," *IEEE Trans. Inf. Theory*, vol. 48, no. 6, pp. 1243–1249, 2002.

[4] H.-I. Su and A. El Gamal, "Two-way source coding through a relay," in *Proc. IEEE Int. Symp. on Inf. Theory (ISIT)*, 2010, pp. 176–180.

[5] F. E. Rezagah and E. Erkip, "Interactive relay assisted source coding," in *Proc. IEEE Global Conference on Signal and Information Processing (GlobalSIP)*. Austin, TX, 2013, pp. 819–822.

[6] L. Vega, P. Piantanida, and A. Hero, "The three-terminal interactive lossy source coding problem," *arXiv preprint*, 2015, arXiv:1502.01359.

[7] Y. Chia, H. Permuter, and T. Weissman, "Cascade, triangular, and two-way source coding with degraded side information at the second user," *IEEE Trans. Inf. Theory*, vol. 58, no. 1, pp. 189–206, 2012.

[8] A. Sgarro, "Source coding with side information at several decoders," *IEEE Trans. Inf. Theory*, vol. 23, no. 2, pp. 179–182, 1977.

[9] S. Li, R. Yeung, and N. Cai, "Linear network coding," *IEEE Trans. Inf. Theory*, vol. 49, no. 2, pp. 371–381, 2003.

[10] D. Slepian and J. Wolf, "Noiseless coding of correlated information sources," *IEEE Trans. Inf. Theory*, vol. 19, no. 4, pp. 471–480, 1973.

[11] A. Wyner and J. Ziv, "The rate-distortion function for source coding with side information at the decoder," *IEEE Trans. Inf. Theory*, vol. 22, no. 1, pp. 1–10, 1976.

[12] T. Berger, "Multiterminal source coding," in *The InformationTheory Approach to Communications*, G. Longo, Ed. New York: Springer-Verlag, 1978.

[13] S.-Y. Tung, "Multiterminal source coding," Ph.D. dissertation, Cornell University, Ithaca, NY, 1978.

[14] M. Gastpar, "The Wyner-Ziv problem with multiple sources," *IEEE Trans. Inf. Theory*, vol. 50, no. 11, pp. 2762–2768, 2004.

[15] R. Timo, T. Chan, and A. Grant, "Rate distortion with side-information at many decoders," *IEEE Trans. Inf. Theory*, vol. 57, no. 8, pp. 5240–5257, 2011.

[16] H. Yamamoto, "Source coding theory for cascade and branching communication systems," *IEEE Trans. Inf. Theory*, vol. 27, no. 3, pp. 299–308, 1981.

[17] D. Vasudevan, C. Tian, and S. Diggavi, "Lossy source coding for a cascade communication system with side-informations," in *Proc. 44th Allerton Conference on Communications, Control, and Computing*. Monticello, IL, 2006.

[18] P. Ishwar and S. Pradhan, "A relay-assisted distributed source coding problem," in *Proc. IEEE Inf. Theory and Appl. Workshop (ITA)*. San Diego, CA, 2008, pp. 136–141.

[19] P. Cuff, H. Su, and A. El Gamal, "Cascade multiterminal source coding," in *Proc. IEEE Int. Symp. on Inf. Theory (ISIT)*. Seoul, Korea, 2009, pp. 1199–1203.

[20] H. Permuter and T. Weissman, "Cascade and Triangular Source Coding With Side Information at the First Two Nodes," *IEEE Trans. Inf. Theory*, vol. 58, no. 6, pp. 3339–3349, 2012.

[21] A. Kimura and T. Uyematsu, "Multiterminal source coding with complementary delivery," in *Proc. IEEE Int. Symp. on Inf. Theory Appl. (ISITA)*. Seoul, Korea, 2006, pp. 189–194.

[22] R. Gray, *Conditional rate-distortion theory*. Information Systems Laboratory, Stanford Electronics Laboratories, 1972.

[23] A. El Gamal and Y.-H. Kim, *Network Information Theory*. Cambridge University Press, UK, 2012.