Nonlinear Programming Based Detectors for Multiuser Systems

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Abstract

Maximum likelihood (ML) detection problems for several multiuser systems result in nonlinear optimization problems with unacceptably high complexity. One way of achieving near-optimum performance without the complexity associated with the ML detector is using nonlinear programming relaxations to approximate the solution of the ML detection problem at hand. Using this approach, new detectors are formulated and it is observed that some popular suboptimum receivers correspond to relaxations of the ML detectors. We concentrate on two types of systems to demonstrate this concept and evaluate the performance of the resulting detectors.

1. Introduction

It is well known that detection of multiple mutually interfering users in a communication system is not an easy task. Having matched filter receivers for each user results in poor error performance in near-far scenarios [13]. In particular, for a CDMA system employing linear modulation, the maximum likelihood (ML) detector to detect the bits of all users has been shown to have exponential complexity in the number of users [12]. Similar conclusions can be made about multiuser systems employing nonlinear modulation [8]. Many suboptimum receivers with near-optimum performance have been developed to decrease the complexity of multiuser detection [4, 5, 10]. These suboptimum receivers have been motivated by several criteria. Among the most popular linear detectors are the decorrelator [4], which suppresses the multiple access interference totally while enhancing the Gaussian noise and the MMSE receiver [5], which minimizes the mean squared error between the filter output and the transmitted bit.

The high complexity of a particular ML detector is a direct consequence of the fact that the associated ML detection problem is a complex discrete optimization problem. For example, a quadratic cost function with a multidimensional binary valued feasible set is encountered for CDMA multiuser detection with BPSK modulation. This is a $0-1$ quadratic program for which there exists no efficient algorithm. The general approach in the presence of such hardship, is to approximate the solution by working on an *easier* problem that can be solved efficiently. The easier problem to be solved is a *relaxation* of the original problem. The relaxed solution is then mapped to the solution set of the original problem, ideally arriving at a near-optimum solution.

In this paper, we review the nonlinear programming methods to arrive at near-optimum multiuser detectors. We concentrate on the two aforementioned systems, i.e., multiuser systems employing linear and nonlinear modulation. We observe that some popular suboptimum detectors are relaxed solutions to the optimum detection problem. This approach also enables us to develop new detectors with near-optimum performance.

2. System Model

Consider a K-user synchronous CDMA system with processing gain N where each user transmits one of M signals. The received signal is given by

$$r(t) = \sum_{k=1}^{K} \gamma_{km_k} s_{km_k}(t) + n(t) \tag{1}$$

where γ_{km_k} and $s_{km_k}(t)$ are the received amplitude (with the appropriate phase information embedded if necessary), and the transmitted signature of the kth user for the m_kth message, and $n(t)$ is the additive white Gaussian noise (AWGN) process with power spectral density σ^2.

We will use the above model for both systems we will consider, namely the CDMA system with linear modulation and nonlinear modulation. In particular, linear modulation corresponds to the special case where

$$s_{km_k}(t) = m_k s_k(t) \qquad \forall k, m_k \tag{2}$$

Let us now consider the discrete time model. It can be obtained by projecting the received signal onto an N-dimensional orthonormal basis, e.g., chip matched filtering. Let m_k be the desired message of the kth user. We define the vector $\mathbf{b}_k = [b_{k1} \cdots b_{kM}]^\top \in F$, where

$$b_{km} = \begin{cases} 1 & m = m_k \\ 0 & \text{otherwise} \end{cases}$$

and $F = \{[1\,0 \cdots 0]^\top, \ldots, [0 \cdots 0\,1]^\top\}$. The $N\times 1$ vector s_{km} represents the chip matched filtered version of the kth user's signature corresponding to message m. The $N\times M$ signature matrix of the kth user is denoted as $S_k \triangleq [s_{k1} \cdots s_{kM}]$; A_{km} and ϕ_{km} represent the amplitude and phase of user k associated with message m, $\gamma_{km} = A_{km} e^{j\phi_{km}}$; $A_k \triangleq \text{diag}[A_{k1}, \ldots, A_{kM}]$ and $\Phi_k \triangleq \text{diag}[e^{j\phi_{k1}}, \ldots, e^{j\phi_{kM}}]$ are $M\times M$ diagonal matrices representing the amplitudes and phases of all the M messages of user k. The received vector can be written as

$$r = \sum_{k=1}^{K} S_k A_k \Phi_k b_k + n \tag{3}$$

where n is the AWGN vector. Further, r can be expressed in terms of the $MK\times 1$ vector $b = [b_1^\top \cdots b_K^\top]^\top$, the $N\times MK$ matrix $S \triangleq [S_1 \cdots S_K]$, the $MK\times MK$ matrices $A \triangleq \text{diag}[A_1, \ldots, A_K]$ and $\Phi \triangleq \text{diag}[\Phi_1, \ldots, \Phi_K]$ as

$$r = SA\Phi b + n \tag{4}$$

In what follows, we concentrate on the two aforementioned systems, observe that using (4), we arrive at joint ML detection problems that are nonlinear programs without efficient solution methods, and investigate approximation methods.

3. MUD with Linear Modulation

Consider, for simplicity, a CDMA system employing BPSK, i.e., $M = 2$ and $s_{kj}(t) = a_k s_k(t)$, $j = 1, 2$. Further, let us consider coherent detection. In this case, (4) can be written in a more compact fashion as

$$r = \bar{S}Aa + n \tag{5}$$

where $\bar{S} \triangleq [s_1 \cdots s_K]$ contains the signatures of the users, A is the diagonal matrix containing the users' received amplitudes, a_k is the bit (± 1 equiprobably) of the kth user.

The chip matched filtered signal given by (5) or equivalently, the vector of matched filter outputs is a sufficient statistic for the multiuser detection problem and is given by

$$y = \bar{S}^\top \bar{S} A a + \bar{n} = \Gamma A a + \bar{n} \tag{6}$$

where Γ is the cross correlation matrix with $\Gamma_{ij} = \int_0^T s_i(t) s_j(t)\,dt$, a is the bit vector of the users and and $\bar{n}$ is a zero mean Gaussian random vector with auto covariance matrix $E[\bar{n}\bar{n}^\top] = \sigma^2 \Gamma$. The ML multiuser detection problem [13] is:

$$a^* = \arg \min_{a \in \{-1,1\}^N} a^\top R a - 2a^\top A y \tag{7}$$

where $R = A\Gamma A$ with $R_{ij} = A_{ii} A_{jj} \Gamma_{ij}$.

Although it has been shown recently that certain special R structures allow construction of polynomial time algorithms to find the optimum solution [9], the problem for general correlation matrices remains NP hard and one can find the optimum a only by exhaustive search of 2^K candidate vectors.

Let us concentrate on cases where the signatures of the users are independent and Γ and hence R are positive definite. In this case, the objective (7) is strictly convex in a and has a well defined unique minimizer over a convex set. Thus, we can find solutions by relaxing the constraint set – which in the original problem contains only the corners of the unit hypercube– such that the resulting "relaxed" constraint set is convex. Note that the requirement is that for each relaxation, the relaxed constraint set contains the feasible set of the original problem. The solution can then be mapped to the feasible set of the original problem by taking the sign of each component of the relaxed solution vector (bits are equiprobably ± 1).

- **Decorrelator**

First consider the simplest relaxation, where the feasible set is relaxed to contain the K dimensional space $\mathcal{R}^K$.

$$\min_{a \in \mathcal{R}^K} a^\top R a - 2a^\top A y \tag{8}$$

This problem has a unique minimum at

$$\hat{a} = R^{-1} A y = a + A^{-1} \Gamma^{-1} \bar{n} \tag{9}$$

Taking the sign of the solution vector yields the well known *decorrelating detector* [4].

- **Soft Interference Cancellation**

The constraint set of the optimum multiuser detection problem (7) consists of the corner points of the unit hypercube. An effective approximation method is to relax the constraint set to cover to whole hypercube and use nonlinear programming algorithms to find the solution of the new convex programming problem [2]. The relaxed problem is:

$$\hat{a} = \arg \min_{a \in [-1,1]^K} a^\top R a - 2a^\top A y \tag{10}$$

Now consider the implementation of the receiver given by (10). Since the optimization is a convex minimization over a convex set, the unique fixed point is the minimum. However, the optimum point does not have a closed form and one should use iterative methods to get to the solution. Fortunately, the simplicity of the constraint set, i.e., the fact that it has a cartesian product form, enables us to define special iterative projection algorithms [1]. In particular, the *nonlinear Gauss-Seidel* and the *nonlinear Jacobi* algorithms respectively, converge to the minimum of (10) under certain conditions. Both algorithms optimize one variable (a_i) at a time to get to the optimum point of (10); however the

Gauss-Seidel method uses the most recent estimates of all variables, while the Jacobi algorithm allows a parallel implementation. It can be shown that the Gauss-Seidel and Jacobi iterations yield the following two-step algorithms. For each user i, the first step for the Gauss-Seidel iteration is, $\hat{x}(n+1) =$

$$\frac{1}{A_{ii}}\left(y_i - \sum_{j=1}^{i-1} A_{jj}\Gamma_{ji}a_j(n+1) - \sum_{j=i+1}^{K} A_{jj}\Gamma_{ji}a_j(n)\right) \tag{11}$$

and the first step for the Jacobi iteration is,

$$\hat{x}(n+1) = \frac{1}{A_{ii}}\left(y_i - \sum_{j=1, j\neq i}^{K} A_{jj}\Gamma_{ji}a_j(n)\right) \tag{12}$$

The second step for both algorithms is

$$a_i(n+1) = \begin{cases} -1, & \hat{x}(n+1) < -1 \\ \hat{x}(n+1), & -1 \leq \hat{x}(n+1) \leq 1 \\ 1, & \hat{x}(n+1) > 1 \end{cases} \tag{13}$$

At each stage, to get the estimate of each user's bit, both receivers use soft estimates of the bits to reconstruct the interference and subtract this estimate from the user's matched filter output, scale the result by the amplitude of the user and project onto $[-1, 1]$. The difference between the two is that while the Gauss-Seidel algorithm uses the available current stage estimates of the users, i.e., feedback from a group of users whose bit estimates are already computed, the Jacobi algorithm uses only bit estimates from the previous stage. Convergence of both the Gauss-Seidel and the Jacobi algorithms can be established using basic nonlinear programming theorems [1]. The convergence conditions depend on the cross correlation matrix $\mathbf{\Gamma}$ and is given in [14].

In general, it takes more than one iteration for either algorithm to converge and thus the resulting receivers are *multistage* receivers. Multi-stage receivers are familiar in multiuser detection. [10] proposes using hard decision bit estimates to reconstruct and subtract the interference for each user. [7] proposes a class of receivers based on the SAGE algorithm, one of which is the successive multistage receiver (11) and argues that the SAGE based hard decision multistage receiver is convergent even when its parallel counterpart (12) is not. The soft decision versions of these multistage receivers, i.e., (11) and (12), are proposed in [7] and [15]. Representing these receivers in the form of iterative nonlinear programming algorithms enables us to observe that *both* receivers, i.e., the parallel and the successive soft multi-stage interference cancellers, if they converge, converge to the same point which is the minimizer of (10).

It is worthwhile to note that, one can implement the decorrelator given by (9) iteratively. Gauss-Seidel and Jacobi algorithms that converge to (9) can be found to be the algorithms derived in this section without the second stage $[-1, 1]$ clippers. The convergence conditions are identical to those discussed in this section. It is also possible to derive Gauss-Seidel and Jacobi iterations that converge to the MMSE detector [5] which estimates the bits by taking the sign of $\bar{\boldsymbol{a}} = (\mathbf{\Gamma} + \sigma^2\boldsymbol{A}^{-2})^{-1}\boldsymbol{y}$. The resulting algorithms differ from (11) and (12) only in the scaling factor [14].

- **Generalized MMSE Detector**

The constraint on each $a_i \in \{-1, 1\}$ is equivalent to $a_i^2 = 1$ which implies $\boldsymbol{a}^T\boldsymbol{a} = K$ at any feasible point for ML-MUD. Relaxing this set to $\boldsymbol{a}^T\boldsymbol{a} \leq K$ results in:

$$\min_{\boldsymbol{a}^\top\boldsymbol{a}\leq K} \boldsymbol{a}^\top\boldsymbol{R}\boldsymbol{a} - 2\boldsymbol{a}^\top\boldsymbol{A}\boldsymbol{y} \tag{14}$$

Since (14) minimizes of a convex function over a convex set, it has a unique minimum and a variety of iterative algorithms can be employed the simplest of which is found by solving the dual problem [1]. The solution becomes

$$\hat{\boldsymbol{a}} = (\boldsymbol{R} + \lambda^*\boldsymbol{I})^{-1}\boldsymbol{A}\boldsymbol{y} = \boldsymbol{A}^{-1}(\mathbf{\Gamma} + \lambda^*\boldsymbol{A}^{-2})^{-1}\boldsymbol{y} \tag{15}$$

with

$$\bar{\lambda}(n+1) = \bar{\lambda}(n) - \mu(\boldsymbol{y}^\top\boldsymbol{A}(\boldsymbol{R} + \bar{\lambda}(n)\boldsymbol{I})^{-2}\boldsymbol{A}\boldsymbol{y} - K)$$

which converges to $\bar{\lambda}$ for a suitable μ, and $\lambda^* = \max(0, \bar{\lambda})$.

The form of this solution whose sign is the estimate of the bit vector is familiar because of its similarity to the *MMSE detector* [5]. We term the relaxation (14) the *generalized MMSE* (GMMSE) solution. When $\lambda^* = \sigma^2$, (15) reduces to the MMSE detector. Note that the GMMSE is a nonlinear multiuser detector in contrast to the MMSE detector.

4. MUD with Nonlinear Modulation

Let us revisit the discrete representation given by (4) and consider nonlinear modulation. In this case, each user transmits one of M signals to transmit one of M symbols [11]. Let us concentrate on cases where the possible waveforms for all messages of all the users are linearly independent.

First consider coherent detection. The ML estimate of $\boldsymbol{b}$ given $\boldsymbol{r}$, $\boldsymbol{A}$ and $\mathbf{\Phi}$ is given by

$$\boldsymbol{b}^* = \arg\min_{\boldsymbol{b}\in\mathcal{F}} \boldsymbol{b}^\top\mathbf{\Phi}^H\boldsymbol{A}\mathbf{\Upsilon}\boldsymbol{A}\mathbf{\Phi}\boldsymbol{b} - 2\text{Re}[\boldsymbol{z}^H\boldsymbol{A}\mathbf{\Phi}\boldsymbol{b}] \tag{16}$$

with $\mathbf{\Upsilon} = \boldsymbol{S}^H\boldsymbol{S}$ and $\boldsymbol{z} = \boldsymbol{S}\boldsymbol{r}$. It is easy to see that the problem above is equivalent to multiuser detection with M-ary linear modulation and that one needs to evaluate all possible sequences (M^K) to find the ML estimate of the messages sent by all users [12]. All nonlinear programming approximations explored for $M = 2$ in the previous section can be easily extended for the case where $M > 2$.

Consider next the case where the amplitudes, $\boldsymbol{A}$, are known at the receiver as in (16), but both $\mathbf{\Phi}$ and $\boldsymbol{b}$ are unknown. In this case, we can estimate $\boldsymbol{x} = \mathbf{\Phi}\boldsymbol{b}$. If we define the set $G = \{[e^{j\phi_1}\ 0 \cdots 0]^\top, \ldots, [0 \cdots 0\ e^{j\phi_M}]^\top\}$, then $\boldsymbol{x} \in \mathcal{G}$, where $\mathcal{G} = G \times G \times \ldots \times G$. The jointly optimum estimate $\boldsymbol{x}^*$ is the solution to

$$\min_{\boldsymbol{x} \in \mathcal{G}} \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{\Upsilon} \boldsymbol{A} \boldsymbol{x} - 2\mathrm{Re}[\boldsymbol{z}^H \boldsymbol{A} \boldsymbol{x}] \qquad (17)$$

The implementation of this detector also requires an exhaustive search and hence has unacceptably high complexity. Specifically, for each possible value of $\boldsymbol{b}$, one has to solve for $\boldsymbol{x}$ which minimizes the objective function in (17) and whose non-zero elements contain the phase information that correspond to the nonzero elements in $\boldsymbol{b}$. Since each nonzero element lies on the unit circle, this minimization is over a *non-convex* set. Thus, we relax the constraints and allow each element to lie *within* the unit circle to guarantee the existence of the unique minimum. Henceforth, we will refer to this detector as the *Joint Detector*.

Similar to their counterparts for linear and coherent detection, near-optimum receivers for nonlinear and noncoherent multiuser detection are also investigated in the literature [3,6,11]. Decorrelative and MMSE detectors that consist of two stages are considered: the pre-filter stage aims at suppressing the interference completely (decorrelator) or minimizing the MSE between the information symbol and the output of the filter (MMSE); the second stage uses one of several decision mechanisms such as maximum magnitude (MM) to estimate the symbol sent by each user in [3].

Let us again consider the nonlinear programming approach. The decorrelator and the MMSE detector are unconstrained relaxations of the ML detection problem (17) like their coherent linear modulation counterparts (see Section 3). Furthermore, it is possible to use the special structure of transmitted signals to investigate other relaxations. To that end, we consider the constrained detectors as outlined below. We also consider multistage soft interference cancellers that are tailored for this system. The motivation of considering the latter is their superior performance well known in the case of coherent linear multiuser detection [7]. In each case, a second decision stage will follow in which we use the MM rule to estimate each symbol.

• Constrained Noncoherent Multiuser Detection

For every $\boldsymbol{x}_k \in G$, $\boldsymbol{x}_k^H \boldsymbol{x}_k = 1$ for all k. If we relax each of these constraints to be $\boldsymbol{x}_k^H \boldsymbol{x}_k \leq 1$, then (17) becomes a convex program whose minimum is unique and can be found using nonlinear programming methods. We will call the resulting detector the *locally constrained detector*. Also, for every $\boldsymbol{x} \in \mathcal{G}$, $\boldsymbol{x}^H \boldsymbol{x} = K$ and once again by relaxing the constraint to $\boldsymbol{x}^H \boldsymbol{x} \leq K$, we arrive at a convex program. The resulting detector in this case is termed the *globally constrained detector* [8]. In both cases, not surprisingly, the solution closely resembles the Generalized MMSE detector of Section 3. Specifically, it can be shown that both detectors are in the form of

$$\hat{\boldsymbol{x}} = \boldsymbol{A}^{-1}(\boldsymbol{\Upsilon} + \boldsymbol{\Lambda}^* \boldsymbol{A}^{-2})^{-1} \boldsymbol{y} \qquad (18)$$

where $\boldsymbol{\Lambda}^*$ is the diagonal matrix whose diagonal elements the appropriate optimal Lagrange multipliers found for the specific detector.

• Noncoherent Soft Interference Cancellation

We now explore noncoherent realizations of the successive and the parallel multistage receivers proposed in [7, 10]. The successive soft interference canceller is analogous to (11). In the first step, at the $(n+1)$st stage, the kth user's ith element is determined as $x_i(n+1) =$

$$\frac{1}{A_i}\left(y_i - \sum_{j=1}^{i-1} \Upsilon_{ij} A_j x_j(n+1) - \sum_{j=i+1}^{MK} \Upsilon_{ij} A_j x_j(n)\right)$$

All the M entries of user k are thus iteratively determined, and then, in the second step, the entry with the maximum magnitude is selected as

$$x_i(n+1) = \begin{cases} x_i(n+1) & if \quad |x_i(n+1)| \geq |x_j(n+1)| \\ 0 & \text{otherwise} \end{cases}$$

This vector estimate is then used by the $(k+1)$st user for estimating its vector, and so on.

Alternatively, the parallel soft interference canceller estimates all the elements of $\boldsymbol{x}_k$ in parallel as $\boldsymbol{x}_k(n+1) =$

$$(\boldsymbol{\Upsilon}_{kk} \boldsymbol{A}_k)^{-1}\left(\boldsymbol{y}_k - \sum_{j=1}^{k-1} \boldsymbol{\Upsilon}_{kj} \boldsymbol{A}_j \boldsymbol{x}_j(n+1) - \sum_{j=k+1}^{K} \boldsymbol{\Upsilon}_{kj} \boldsymbol{A}_j \boldsymbol{x}_j(n)\right)$$

where $\boldsymbol{\Upsilon}_{kj}$ and $\boldsymbol{A}_k$ are $M \times M$ block matrices. In the second step, the users' messages are obtained by using the same mapping as in successive soft interference canceller.

5. Results and Discussion

We have simulated the bit error performance of the detectors investigated in this work. Figure 1 shows the probability of bit error for the desired user when the system employs BPSK and coherent detection. All iterative detectors (multistage soft cancellers and the GMMSE) are evaluated at their convergence points. The soft interference cancellers ((11), (12), (13)) have almost invariable performance versus interference strength. We note that the performance of the GMMSE detector is similar to that of the linear MMSE detector. In particular, we observe that the GMMSE detector has the same trend of approaching the decorrelator performance as the MMSE detector as the interference dominates the noise. Figure 2 shows the probability of symbol error for one user when an interferer exists and when the system employs noncoherent detection and nonlinear modulation with $M = 4$. The constrained detectors perform similarly with

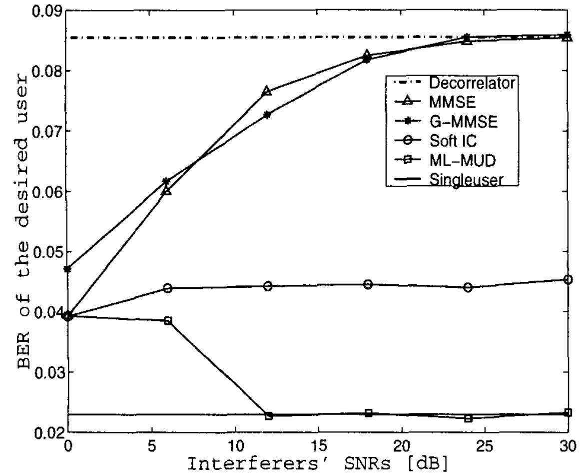

Figure 1. Coherent detection for BPSK. Desired user at 6 dB SNR. $N = 7$, $K = 4$, Gold sequences.

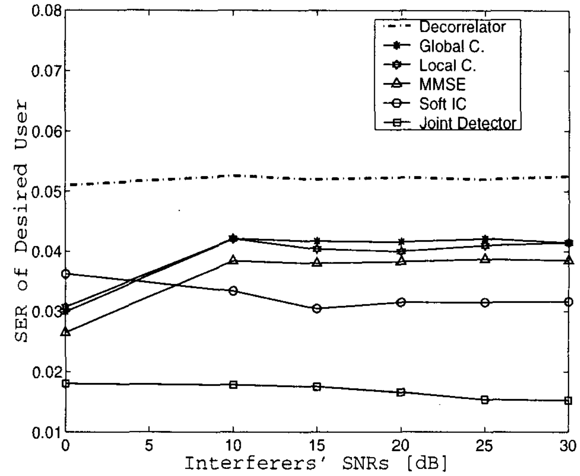

Figure 2. Noncoherent detection for nonlinear modulation. Desired user at 10 dB SNR. $N = 20$, $K = 2$, $M = 4$, random sequences.

the MMSE detector. The 3-stage successive soft interference canceller with decorrelative first stage is observed to perform well especially in the near-far situation. Note that the MMSE detector does not converge to the decorrelator in the high interferer power region in contrast with multiuser systems employing BPSK. This is a direct consequence of the fact that, in the near-far situation, the MMSE detector zero-forces the high power interferers only, not the relatively low power $M - 1$ undesired messages of the desired user.

In this paper, we have shown that many popular suboptimum detectors are devices that attempt to approximate the solution of the joint maximum likelihood multiuser detector for several types of systems. Although it is analytically hard to characterize exactly how closely they approximate the ML cost function, we have observed that they achieve near-optimum cost values. Consequently, the near-optimum bit error rate performances of these detectors are not surprising. The approach helps us identify convergence conditions of multistage detectors and motivates new detectors such as the constrained detectors.

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