Rank Constrained Temporal-Spatial Filters for CDMA Systems in Multipath Channels

Aylin Yener and Önder Filiz
Department of Electrical Engr.
The Pennsylvania State University
121 Electrical Engr. East
University Park, Pennsylvania 16802
Email: {yener,ouf101}@ee.psu.edu

***Abstract*— In this paper, a receiver structure which combines multiuser detection (temporal filtering) and receiver beamforming (spatial filtering) in a multipath environment is considered. Following [1], [2], we model the receiver as a linear matrix filter and use the minimum mean-squared error (MMSE) as the performance criterion. Motivated by the complexity of the optimum receiver, we propose rank constrained temporal-spatial filters which are simpler and near optimum. The MSE is minimized subject to a structural constraint, using an iterative alternating minimization algorithm. Numerical results indicate that a performance close to that of the optimum filter can be achieved with a simple iterative structure, even in highly loaded systems. Least mean squares (LMS) is used to formulate the adaptive implementations and the convergence properties are investigated along with the effect of the number of multipath components.**

I. INTRODUCTION

CDMA systems suffer from multi-access interference (MAI), which degrades performance. This paper considers combined multiuser detection (temporal filtering) and receiver beamforming (spatial filtering) as the MAI suppression mechanism. Multiuser detection [3] exploits the inherent structure of multiple access interference to suppress it effectively. One of the near optimum detectors is the MMSE detector, designed to minimize the mean-squared error between the filter output and the information bit. Adaptive implementations of the MMSE detector based on training bits as well as blind versions were studied in [4], [5]. Antenna arrays have long been utilized to increase the capacity of wireless systems: beamforming separates the desired signal from interfering signals that originate from different locations [6]. The capacity increase achieved with base-station antenna arrays in CDMA systems is shown in [7], where matched filters are assumed for each user.

Combined temporal-spatial filtering was considered in [2] and the necessary statistics were derived along with several multiuser detectors. A recent paper examines several two dimensional MMSE linear filter structures in a single path environment [8]. The complexity of the joint optimum MMSE temporal-spatial filter (OTSF) led the authors to propose rank-1 constrained filters that employ the same temporal multiuser detector at each antenna element. Rank-1 constrained filter's simplicity is appealing from an implementation point of view, but the performance of such filters may be severely suboptimal compared to the optimal filter due to the constrained solution space, especially under heavy loads.

Motivated by the performance gap between the OTSF and the rank-1 constrained filter, in this work, we search for filter structures whose performance lie between that of OTSF and the rank-1 constrained filter and which can exploit the diversity provided by the multipath environment. We propose a general class of rank constrained filters, which are found subject to a structural constraint on the receiver filter. The constraint can be relaxed in order to achieve a near optimum performance, at the expense of additional complexity.

Following [1], [2], we express the temporal-spatial filter of a user as a receiver matrix filter, which performs joint multiuser detection, receiver beamforming and multipath combining. The MSE, with the appropriate rank constraint, is iteratively minimized using an alternating minimization algorithm. We first derive the deterministic iterations, which assume the knowledge of all users' parameters. Motivated by the fact that adaptive implementations require little information to track the constantly changing parameters in a wireless environment, we consider such implementations in Section 4. An adaptive algorithm which is a combination of the alternating minimization and the least mean squares (LMS) is formulated and the parameters that affect the convergence are explained.

II. SYSTEM MODEL

We consider the uplink of a single cell DS-CDMA system in a multipath environment with K active users and the processing gain is N. An antenna array of M elements is employed at the receiver. The transmitted signal of user k is given by:

$$z_k(t) = \sqrt{P_k} \sum_{n=-\infty}^{\infty} b_k(n) s_k(t - nT_s) \quad (1)$$

where P_k, b_k and $s_k(t)$ represent the transmit power, information bit and the signature waveform of user k. The temporal signatures of the users are of the form:

$$s_k(t) = \sum_{n=1}^{N} s_k[n]\, \phi(t-(n-1)T_c), \quad s_k[n] = \pm\frac{1}{\sqrt{N}} \quad (2)$$

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where $\phi(t)$ is the unit energy chip waveform and T_c is the chip duration. The transmitted signal passes through a multipath channel with impulse response given by:

$$h_k(t) = \sum_{l=1}^{L} h_{k,l}\,\delta(t - \tau_{k,l}) \tag{3}$$

where $h_{k,l}$ and $\tau_{k,l}$ are the complex channel coefficient and the delay associated with path l of user k. For simplicity, each user's channel is assumed to be composed of exactly L paths and the delays are assumed to be chip synchronous i.e., $\tau_{k,l} = (l-1)T_c$. The received signal at the output of the antenna array is given by:

$$\mathbf{r}(t) = \sum_{k=1}^{K}\sum_{l=1}^{L} h_{k,l}\, z_k(t - \tau_{k,l} - \nu_k)\, \mathbf{a}_{k,l} + \mathbf{n}(t) \tag{4}$$

where ν_k is the delay of user k and $\mathbf{a}_{k,l}$ is the array response vector of path l of user k. If a linear antenna array with $\lambda/2$ spacing is employed, the m^{th} element of $\mathbf{a}_{kl}$ is given by $\mathbf{a}_{k,l}[m] = exp\left(-j\pi(m-1)cos(\theta_{k,l})\right)$, where $\theta_{k,l}$ is the direction of arrival (DOA) of path l of user k. The received signal, $\mathbf{r}(t)$ is chip matched filtered and sampled over the entire observation interval of $(N + L - 1)$ chips, with the receiver synchronized to the first path of the desired user such that $\tau_{i,1} = 0$. The resulting data for the n^{th} bit period of the desired user, can be arranged in a $(N+L-1)\times M$ dimensional matrix $\mathbf{R}$:

$$\begin{aligned}\mathbf{R}(n) = &\sum_{k=1}^{K} \sqrt{P_k}\Big[b_k(n-1)\mathbf{S}_k^{(-1)} + b_k(n)\mathbf{S}_k \\ &+ b_k(n+1)\mathbf{S}_k^{(+1)}\Big]\mathbf{H}_k\mathbf{A}_k^T + \mathbf{N}\end{aligned} \tag{5}$$

$\mathbf{H}_k = \text{diag}\left[h_{k,1}, \ldots, h_{k,l}\right]$ and $\mathbf{A}_k = [\mathbf{a}_{k,1}, \cdots, \mathbf{a}_{k,L}]$ are the matrices of channel coefficients and array responses of user k. $\mathbf{S}_k^{(-1)}$, $\mathbf{S}_k$ and $\mathbf{S}_k^{(+1)}$ are the $(N+L-1)\times L$ dimensional delayed signature sequence matrices. Despite being the most general model with asynchronous users and ISI, (5) adds little insight in terms of the performance and complicates the derivations. Therefore, following references [9], [10], we will assume synchronous users (s.t. $\nu_k = 0\ \forall k$) and that $\tau_{k,l} << T_s$ such that ISI can be ignored in the sequel. Hence, the received signal over the observation interval becomes:

$$\mathbf{R} = \sum_{k=1}^{K} \sqrt{P_k} b_k \mathbf{S}_k \mathbf{H}_k \mathbf{A}_k^T + \mathbf{N} \tag{6}$$

where the $(N+L-1)\times L$ signature sequence matrix of user k is given by:

$$\mathbf{S}_k = \begin{bmatrix} s_k[1] & & \mathbf{0} \\ & \ddots & \\ \vdots & & s_k[1] \\ s_k[N] & & \vdots \\ & \ddots & \\ \mathbf{0} & & s_k[N] \end{bmatrix} \tag{7}$$

The $\mathbf{N}$ matrix represents the spatially and temporally white noise, $E[N_{kl}^* N_{mn}] = \sigma^2\delta_{km}\delta_{ln}$ where $(\cdot)^*$ denotes the conjugate operation. A linear matrix filter $\mathbf{X}_i$ is used to compute the decision statistic y_i and the bit decision is made by taking the sign of the real part of y_i:

$$y_i = \sum_{n=1}^{N+L-1}\sum_{m=1}^{M} [X_i]_{nm}^* R_{nm} = tr(\mathbf{X}_i^H\mathbf{R}) \tag{8}$$

where $tr(\cdot)$ and $(\cdot)^H$ denote the trace and hermitian operations respectively.

III. Rank Constrained Temporal-Spatial Filters

The optimum temporal-spatial filter ($\bar{\mathbf{X}}_i$) minimizes the mean-squared error between the decision statistic and the information bit:

$$\bar{\mathbf{X}}_i = \arg\min_{\mathbf{X}} E\left[\left|tr\left(\mathbf{X}^H\mathbf{R}\right) - b_i\right|^2\right] \tag{9}$$

After reformulating the optimization problem with vector variables, the solution is given by [1], [4], [8]:

$$\bar{\mathbf{x}}_i = \sqrt{P_i}\left(\sum_{k=1}^{K} P_k\mathbf{q}_k\mathbf{q}_k^H + \sigma^2\mathbf{I}\right)^{-1}\mathbf{q}_i \tag{10}$$

where $\mathbf{q}_k$ is the long vector obtained by stacking up columns of $\mathbf{S}_k\mathbf{H}_k\mathbf{A}_k^T$. To reconstruct $\mathbf{X}_i$, we take every $(N+L-1)$ element of $\mathbf{x}_i$ and place as a column to $\mathbf{X}_i$. From (10), it is seen that finding the optimum filter requires the inversion of an $(N+L-1)M \times (N+L-1)M$ dimensional matrix which can be computationally costly. Motivated by this complexity of OTSF, the authors of [1] proposed a simpler receiver. In this case, the filter space i.e., the solution space of the optimization problem, is constrained to contain filters of rank 1 only:

$$\begin{aligned}\mathbf{X}_i = &\arg\min_{\mathbf{X}} E\left[\left|tr\left(\mathbf{X}^H\mathbf{R}\right) - b_i\right|^2\right] \\ &s.t. \quad \text{rank}(\mathbf{X}) = 1\end{aligned} \tag{11}$$

Because of the tight constraint on the solution space, rank-1 filter's performance is suboptimal as compared to the OTSF's. This performance difference could be quite pronounced in heavily loaded systems: as the number of interfering users increase with respect to the dimensions provided by the temporal and spatial domains, the solutions found in the constrained space of rank-1 filters become more inadequate. Under such conditions, filters whose performance lie between that of OTSF and the rank-1 constrained filter may be desirable. We propose to achieve this performance increase by replacing the rank constraint with a looser version. By relaxing the constraint, the solution space will expand (including the matrices of rank 1) and the filters found in this new larger space will possibly perform better. Here we will investigate the general class of rank-r constrained filters, where $1 \le r \le \min\{N+L-1, M\}$. In other words, the solution space of the optimization problem will be the space of up to rank r matrices in $C^{N+L-1\times M}$. Note that any matrix filter whose rank is less than or equal to r can be expressed in terms of at most r temporal-spatial filter pairs:

$$\mathbf{X}_i = \sum_{j=1}^{r} \mathbf{c}_{ij} \mathbf{w}_{ij}^T \tag{12}$$

where $\mathbf{c}_{ij}$ and $\mathbf{w}_{ij}$ are $(N+L-1)$ and M dimensional vectors respectively. With this new representation, the decision statistic and the MSE can be expressed as:

$$y_i = \sum_{j=1}^{r} \mathbf{c}_{ij}^H \mathbf{R} \mathbf{w}_{ij}^* \tag{13}$$

$$\begin{aligned} \text{MSE} = &\sum_{l=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{K} P_k \mathbf{c}_{il}^H \mathbf{Q}_k \mathbf{w}_{il}^* \mathbf{w}_{ij}^T \mathbf{Q}_k^H \mathbf{c}_{ij} \\ &+ \sigma^2 \sum_{l=1}^{r} \sum_{j=1}^{r} \left(\mathbf{c}_{il}^H \mathbf{c}_{ij} \right) \left(\mathbf{w}_{il}^H \mathbf{w}_{ij} \right) \\ &- 2 \sum_{j=1}^{r} \sqrt{P_i} \Re \left\{ \mathbf{c}_{ij}^H \mathbf{Q}_i \mathbf{w}_{ij}^* \right\} + 1 \end{aligned} \tag{14}$$

where $\mathbf{Q}_k = \mathbf{S}_k \mathbf{H}_k \mathbf{A}_k^T$. A closed form expression for the minimizer of the MSE does not exist. In addition, the MSE is not jointly convex in all vector variables. However, it is convex for a single variable given the remaining $2r-1$ variables are fixed. An alternating minimization based iterative algorithm can be used to iteratively minimize the MSE [11]. Each step of the algorithm consists of $2r$ sub-steps, and at each sub-step of the algorithm, a single variable is updated to minimize the MSE, while the remaining $2r-1$ variables are fixed. At the next sub-step a different variable is updated and this procedure continues in a round-robin fashion until convergence. With some abuse of notation let MMSE$(\{\mathbf{c}_{ij}\}_{j \neq x}, \{\mathbf{w}_{ij}\}_{j=1}^r)$ and MMSE$(\{\mathbf{c}_{ij}\}_{j=1}^r, \{\mathbf{w}_{ij}\}_{j \neq x})$ denote values of $\mathbf{c}_{ix}$ and $\mathbf{w}_{ix}$ that minimize MSE given that the remaining $2r-1$ variables are fixed. After setting the gradient equal to zero and solving for vector variables, the following equations are found:

$$\begin{aligned} \hat{\mathbf{c}}_{ix} = &\text{MMSE}(\{\mathbf{c}_{ij}\}_{j \neq x}, \{\mathbf{w}_{ij}\}_{j=1}^r) \\ = &\left(\sum_{k=1}^{K} P_k \mathbf{Q}_k \mathbf{w}_{ix}^* \mathbf{w}_{ix}^T \mathbf{Q}_k^H + \sigma^2 |\mathbf{w}_{ix}|^2 \mathbf{I} \right)^{-1} \\ \times &\left(\sqrt{P_i} \mathbf{Q}_i \mathbf{w}_{ix}^* - \sum_{j \neq x}^{r} \left(\sum_{k=1}^{K} P_k \mathbf{Q}_k \mathbf{w}_{ix}^* \mathbf{w}_{ij}^T \mathbf{Q}_k^H + \sigma^2 \mathbf{w}_{ix}^H \mathbf{w}_{ij} \mathbf{I} \right) \mathbf{c}_{ij} \right) \end{aligned} \tag{15}$$

$$\begin{aligned} \hat{\mathbf{w}}_{ix} = &\text{MMSE}(\{\mathbf{c}_{ij}\}_{j=1}^r, \{\mathbf{w}_{ij}\}_{j \neq x}) \\ = &\left(\sum_{k=1}^{K} P_k \mathbf{Q}_k^T \mathbf{c}_{ix}^* \mathbf{c}_{ix}^T \mathbf{Q}_k^* + \sigma^2 |\mathbf{c}_{ix}|^2 \mathbf{I} \right)^{-1} \\ &\left(\sqrt{P_i} \mathbf{Q}_i^T \mathbf{c}_{ix}^* - \sum_{j \neq x}^{r} \left(\sum_{k=1}^{K} P_k \mathbf{Q}_k^T \mathbf{c}_{ix}^* \mathbf{c}_{ij}^T \mathbf{Q}_k^* + \sigma^2 \mathbf{c}_{ix}^H \mathbf{c}_{ij} \mathbf{I} \right) \mathbf{w}_{ij} \right) \end{aligned} \tag{16}$$

A summary of this algorithm is shown in Table 1.

```
FOR   t = 1 : T
    FOR   x = 1 : r
        ĉ_ix = MMSE({c_ij}_{j≠x}, {w_ij}_{j=1}^r)
        ŵ_ix = MMSE({c_ij}_{j=1}^r, {w_ij}_{j≠x})
    END
END
```

TABLE I

SUMMARY OF THE ALTERNATING MINIMIZATION ALGORITHM FOR THE RANK r CASE, WHERE T IS THE TOTAL NUMBER OF ITERATIONS

IV. ADAPTIVE IMPLEMENTATIONS

In this section, adaptive implementations of the rank constrained filters are formulated. The deterministic case assumes the knowledge of all users' temporal signatures, array response vectors, transmit powers and channel information. Not all these parameters may be available to the system, especially in a cellular scenario. Adaptive implementations are preferred because the only side information required is the desired user's training bits.

The adaptive implementation that we propose here is a combination of the alternating minimization approach of the previous section and the least mean squares (LMS) algorithm. While keeping the main structure of the alternating minimization algorithm, each sub-step will be treated as an independent LMS problem. In the deterministic case, at each sub-step, one variable is updated to maximally decrease MSE. LMS on the other hand is a recursive method that uses noisy estimates of the gradient to update the filter estimate along the direction of the steepest descent. Because of the stochastic nature of LMS, in principle infinite iterations are required to reach the optimal point, whereas in the deterministic case, the same is accomplished with a single update (Equations (15) and (16)). Since it is not feasible to wait for such long periods, for each sub-step, we will only use B training bits. When the algorithm moves on to the next sub-step, the MSE function (error surface) will be changed and a new LMS algorithm will begin. Therefore, the same set of training bits could be reused, for a more efficient use of the resources, bandwidth and time.

The classic update rule of LMS is given by [12]:

$$\mathbf{w}_i(n+1) = \mathbf{w}_i(n) + \mu \left[b_i(n) - y_i(n) \right]^* \mathbf{u}(n) \tag{17}$$

where $\mathbf{w}_i(n)$, μ, $b_i(n)$, $y_i(n)$ and $\mathbf{u}(n)$ represent the filter estimate, step size, the desired response, decision statistic and the received signal respectively. Applying this rule to our case results in the following equations:

$$\mathbf{c}_{ix}(n+1) = \mathbf{c}_{ix}(n) + \mu \left[b_i(n) - y_i(n) \right]^* \mathbf{R}(n) \mathbf{w}_{ix}^*(n) \tag{18}$$

$$\mathbf{w}_{ix}(n+1) = \mathbf{w}_{ix}(n) + \mu \left[b_i(n) - y_i(n) \right]^* \mathbf{R}^T(n) \mathbf{c}_{ix}^*(n) \tag{19}$$

Step size is an important parameter in LMS algorithms. When a small step size is used, the algorithm converges slower but eventually it achieves better performance (higher SIR in this case) with respect to a larger step size. We expect our algorithm to exhibit the same characteristics. Although in principle an infinite number of iterations are required to reach the optimal point in the LMS algorithm, in our implementation we limit the number of recursive iterations to B. The value of B should be chosen large enough to avoid premature jumping to the next step, but small enough to avoid unacceptably slow convergence of the overall adaptive algorithm.

V. Numerical Results

We consider a single cell CDMA system that employs a base station antenna array. Signature sequences, direction of arrivals (DOA) and channel coefficients are randomly generated for each run. The results are time averages of 100 runs. We assume that both the signature sequences and array responses are of unit energy. Channel coefficients are zero mean complex Gaussian variables, normalized such that $E[|h_{k,l}|^2] = 1$. Note that with appropriate scaling, MSE and SIR produced by every linear matrix filter can be related [8].

We first consider a system with processing gain $N = 16$ and $M = 8$ array elements. There are $K = 40$ users, each with $L = 3$ paths and interfering users' powers are set to 10 dB higher than the desired user. Figure 1 shows the output SIR. It is seen that each iteration increases the SIR monotonically. As expected, filters with higher maximum allowable rank perform better: rank-4 filter converges faster and achieves a higher SIR compared to the rank-1 filter. It is seen that near optimum performance can be achieved with rank-4 filters without compromising the simple iterative structure of rank constrained filters.

Note that the rank constrained temporal-spatial filters are different than the reduced-rank methods in the sense that the term 'rank' in our case refers to the rank of the receiver matrix filter, whereas in the latter case, it refers to the rank of the covariance matrix of the received signal vector [13]. For the system described in Figure 1, the output SIR of rank constrained filters are compared with the reduced-rank multistage wiener filter (MSWF). The MSWF curve is generated by formulating equation (9) with long vector variables and applying the MSWF techniques described in [13], [14]. Figure 2 shows that the near optimum performance of our rank-4 filter is achieved by a reduced-rank filter of dimension 10.

The effect of the number of paths on the performance is shown in Figure 3, where maximum SIR achieved at the convergence point of the algorithm, is plotted versus the number of paths. Since increasing the number of paths has the effect of increasing the effective temporal signature length, the proposed filters typically perform better with multipath diversity.

For the rest of this section, the system considered has $M = 8$ antenna array elements, $N = 16$ processing gain and $K = 40$ equal power users, where each user has $L = 3$ paths. Figure 4 compares the adaptive implementations of different rank constrained filters. Blocks of $B = 10$ training bits are used at each sub-step of the alternating minimization algorithm such that each block of 10 bits corresponds to one step (iteration index) of the deterministic case. Similar to Figure 1, filters with higher maximum allowable rank perform better.

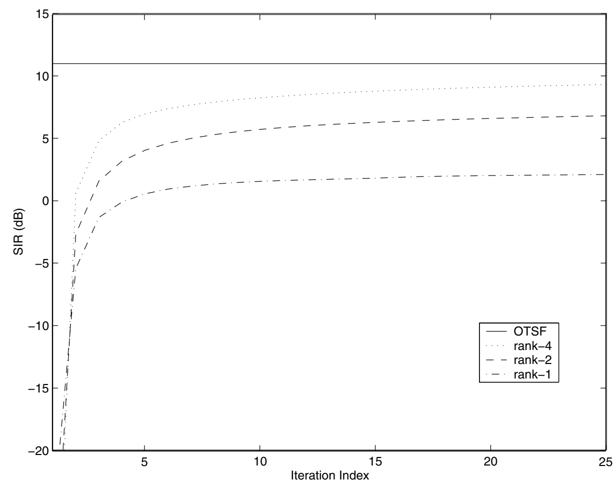

Fig. 1. $K = 40,\ N = 16,\ M = 8,\ L = 3$

For this system, the effect of μ is observed in Figure 5; where an increase in the step size increases the convergence speed, but degrades the performance. Figure 6 compares the behavior of rank-4 filters in different environments and it is seen that the maximum achieved SIR increases as the number of paths increase.

VI. Conclusion

In this paper, we have proposed the rank constrained filters and derived their adaptive counterparts in multipath channels. It is shown that with looser rank constraint, better performance can be achieved at the expense of additional complexity. Even in heavily loaded systems, where there is a significant performance gap between OTSF and rank-1 filter, near full-rank performance can be achieved with a mild increase in complexity with respect to the rank-1 constrained filter. Adaptive implementations based on LMS are also formulated. The existence of multipath provides diversity and by appropriate combining included in our filter structures, we are able to improve performance, in both the deterministic and the adaptive cases. To further enhance the interference management capacity of the system, rank constrained filters could be combined with power control algorithms [1].

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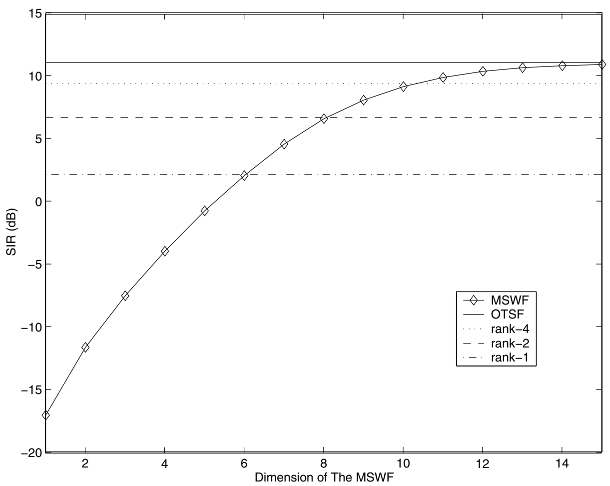

Fig. 2. MSWF and rank constrained filters

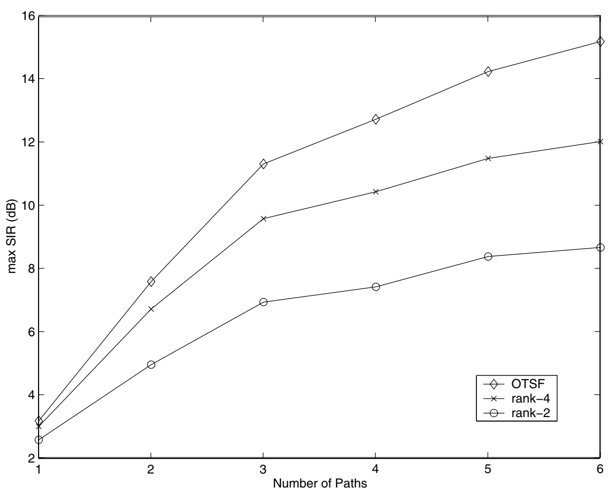

Fig. 3. $K = 10,\ N = 16,\ M = 8,\ L = 3$

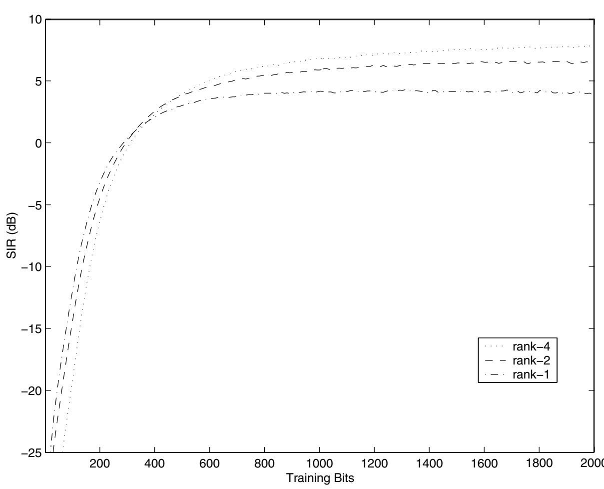

Fig. 4. $K = 40,\ N = 16,\ M = 8,\ L = 3,\ \mu = 0.01$

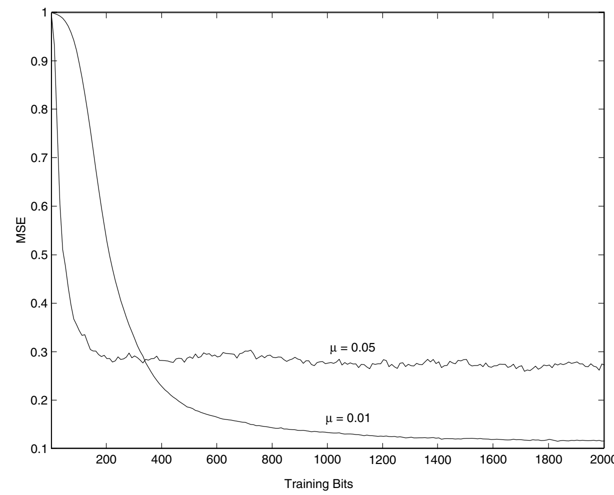

Fig. 5. $K = 40,\ N = 16,\ M = 8,\ L = 3,\ \text{rank}(\mathbf{X}) = 2$

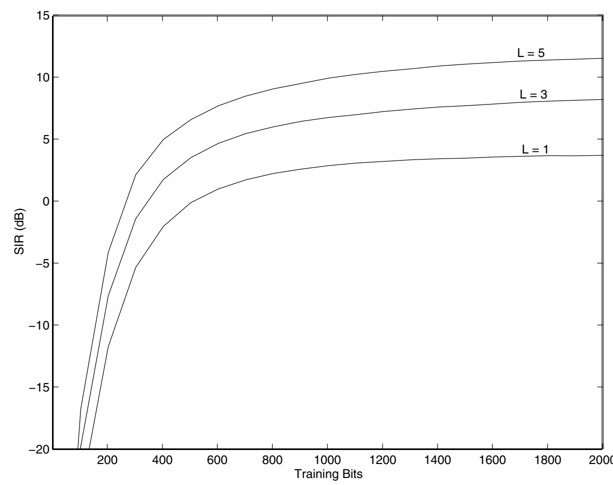

Fig. 6. $K = 40,\ N = 16,\ M = 8,\ \mu = 0.01,\ \text{rank}(\mathbf{X}) = 4$

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